Representations of Quantum Group at Roots of Unity

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Notations

- $\mathfrak{g}$: complex semi-simple Lie algebra
- (a_{ij}) : Cartan matrix
- d_i : coprime integers s.t. $(d_i a_{ij})$ is symmetric
- $q_i := q^{d_i}$
- $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$
- $U_{\mathcal{A}} :=$ $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $(E_i)^{(r)}$, $(F_i)^{(r)}$, and $K_i^{\pm 1}$, $1 \leq i \leq n$
- $U_{\mathcal{A}}^{+} :=$ $\mathcal{A}$-subalgebra of $U_{\mathcal{A}}$ generated by $(E_i)^{(r)}$, $i = 1, 2, \ldots, n$, $r \in \mathbb{N}$
- $U_{\mathcal{A}}^{-} :=$ (——)
  $(F_i)^{(r)}$ $\longmapsto$

...

- $\mathcal{U}^0_{\mathcal{A}} :=$ $\langle \ \rangle$ $K_i^{\pm 1}$, and $\begin{bmatrix} K_i ; c \\ r \end{bmatrix}_{q_i}$, $\mapsto$

- $\mathcal{U}_{\epsilon,\mathbb{Z}} := \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Z}[\epsilon]$, $\epsilon \in \mathbb{C}^\times$, a primitive l-th root of unity, l is odd, $l > d_i$, $\forall i$.

- $\mathcal{U}_\epsilon := \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{Q}(\epsilon)$

- $\epsilon_i := \epsilon^{d_i}$

Reference

- A Guide to Quantum Groups   Chari & Pressley
- Repns of Quantum Algebras   Anderson, Polo, & Wen

§. Repns of U_ϵ.

Let V be a repn. of U_ϵ.

Ivan's talk $\Rightarrow$

$K_1, \cdots, K_n$ act on V via commuting operators which are

- simultaneously diag.
- have eigenvalues in $\{\pm\epsilon^r\}_{r=0,1,\dots,l-1}$.

Tensoring a 1-dim repn. allows us to restrict our attention to repns. on which K_i^l acts by 1, $\forall i$. Call such repns. of type 1.

<u>Lemma</u> Let V be a finite-dimensional U_ϵ-repn. of type 1. Then

(i) $(E_i)^{(l)}$ and $(F_i)^{(l)}$ act nilpotently on V.

(ii) $\begin{bmatrix} K_i; 0 \\ l \end{bmatrix}_{\epsilon_i}$ act on V diag w/ integer eigenvalues.

<u>Proof</u> Exercise.

Hint:

- $\begin{bmatrix} K_i; c \\ t \end{bmatrix} E_j^{(s)} = E_j^{(s)} \begin{bmatrix} K_i; c+sa_{ij} \\ t \end{bmatrix}$

- $\begin{bmatrix} K_i; c \\ t \end{bmatrix} F_j^{(s)} = F_j^{(s)} \begin{bmatrix} K_i; c-sa_{ij} \\ t \end{bmatrix}$

- $(E_i)^{(r)}(F_j)^{(s)} = \sum_{r,s\geq t\geq 0} (F_i)^{(s-t)} \begin{bmatrix} K_i; \; 2t-s-r \\ t \end{bmatrix}_{\epsilon_i} (E_i)$

- $\begin{bmatrix} K_i;0 \\ \ell r \end{bmatrix}_\epsilon = \frac{1}{r!} \prod_{s=0}^{r-1} \left(\begin{bmatrix} K_i;0 \\ \ell \end{bmatrix}_\epsilon - s \right).$

- <u>Def</u>. Let V be a type 1 repn. We define the weight space

$$V_\lambda := \left\{ v \in V \;\middle|\; K_i \cdot v = \epsilon_i^{(\lambda, \alpha_i^\vee)} v, \; \begin{bmatrix} K_i;0 \\ \ell \end{bmatrix}_{\epsilon_i} \cdot v = \begin{bmatrix} (\lambda, \alpha_i^\vee) \\ \ell \end{bmatrix}_{\epsilon_i} \cdot v, \; i = 1, \dots, n \right\}$$

for $\lambda \in \Lambda$.

If V is finite-dimensional.

$$ch(V) = \sum_{\lambda \in \Lambda} \dim(V_\lambda) e^\lambda.$$

- <u>Prop</u>. Assume V is a type 1 repn. of U_ϵ. Then

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

<u>Proof</u> Exercise. Hint:

<u>Lemma</u>. Let ϵ: primitive ℓ-th root of 1, ℓ odd, and $r, s \in \mathbb{N}$. Assume $r = r_0 + \ell r_1$, $s = s_0 + \ell s_1$, $0 \leq r_0, s_0 < \ell$, $r_1, s_1 \geq 0 \Rightarrow$

$$\begin{bmatrix} r \\ s \end{bmatrix}_\epsilon = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}_\epsilon \cdot \binom{r_1}{s_1}.$$

<u>Def</u>. Let V be a repn. of U_ϵ of type 1. A vector $v \in V$ is called primitive if $E_i \cdot v = E_i^{(\ell)} v = 0$, $\forall i = 1, \dots, n$.

Call V is a highest weight module if it is generated by a primitive weight vector v_λ.

<u>Lemma</u>. If V is a highest weight module w/ highest weight λ, then $V \cong \bigoplus_{\mu \le \lambda} V_\mu$, and $V_\lambda \cong \mathbb{C} \cdot v_\lambda$.

· <u>Def</u>. The Verma module $\Delta_q(\lambda)$ is defined to
$$U_q / U_q \langle E_i, K_i - q^{\lambda_i} \mid i = 1, \dots, n \rangle,$$ where $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$.

$\Delta_q(\lambda)$ is a free U_q^--module generated by v_λ, and it has a a unique irred quotient $L_q(\lambda)$ w/ highest weight λ.

Let $V_{\mathcal{A}}(\lambda)$ be the $U_{\mathcal{A}}$-submodule of $L_q(\lambda)$ generated v_λ.

· <u>Def</u> (Weyl modules). Let $\lambda \in \Lambda^+$, define the Weyl module
$$W_\epsilon(\lambda) := V_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{C}$$ via the homomorphism
$\mathcal{A} \to \mathbb{C} \ : \ q \mapsto \epsilon$.

· <u>Prop</u>. For any $\lambda \in \Lambda^+$,
(i) $\dim(W_\epsilon(\lambda)) < \infty$
(ii) $\mathrm{ch}(W_\epsilon(\lambda))$ is given by the Weyl character formula.

Since $W_\epsilon(\lambda)$ is a highest weight module, it has a unique irred quotient $L_\epsilon(\lambda)$.

__Cor__. For any $\lambda \in \Lambda^+$, $\dim(L_\epsilon(\lambda)) < \infty$.

__Prop__ Let $\lambda \in \Lambda^+$. If λ satisfies one of the following conditions

(a) $(\lambda + \rho, \alpha^\vee) < \ell \quad \forall\, \alpha \in \Phi^+$

(b) $\lambda = (\ell - 1)\rho + \ell\mu$ for some $\mu \in \Lambda^+$,

then $W_\epsilon(\lambda)$ is irred and $ch(L_\epsilon(\lambda))$ is given by the Weyl character formula.

• __Thm__. Every finite-dimensional irred U_ϵ-module of type 1 is isomorphic to $L_\epsilon(\lambda)$, for some $\lambda \in \Lambda^+$.

__Proof__ Similar to the classical case.

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda \quad .$$

Let λ the maximal weight. Then $0 \neq v_\lambda \in V_\lambda$ is primitive $\Rightarrow$ v_λ generates V.

Remains to show $\lambda \in \Lambda^+$ i.e. $(\lambda, \alpha_i^\vee) \geq 0$.

__Exercise__. Hint • $(F_i)^{(m\ell)} = \frac{1}{m!}\left((F_i)^{(\ell)}\right)^m$

$$\bullet \begin{bmatrix} r \\ s \end{bmatrix}_\epsilon = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}_\epsilon \binom{r_1}{s_1}.$$

We can also define $L_\epsilon(\lambda)$ as follows:

$$\Delta_\epsilon(\lambda) = U_\epsilon \Big/ U_\epsilon \left\langle K_i - \epsilon_i^{(\lambda, \alpha_i^\vee)},\ \begin{bmatrix} K_i ; 0 \\ \ell \end{bmatrix}_{\epsilon_i} - \begin{bmatrix} (\lambda, \alpha_i^\vee) \\ \ell \end{bmatrix}_{\epsilon_i},\ E_i,\ E_i^{(\ell)} \right\rangle.$$

Then $\Delta_\epsilon(\lambda)$ is a highest weight U_ϵ-mod, and $L_\epsilon(\lambda) :=$ unique irred quotient of $\Delta_\epsilon(\lambda)$ of highest wt. λ.

§. Tensor Product Theorem

For any $\lambda \in \Lambda^+$, $\exists!$ $\lambda_0, \lambda_1 \in \Lambda^+$ s.t.

(i) $\lambda = \lambda_0 + l\lambda_1$

(ii) $0 \le (\lambda_0, \alpha_i^\vee) < l$, $\forall i = 1, \dots, n$

- Thm. Let λ, λ_0, and λ_1 be as above. There is an isomorphism of U_ϵ-modules

$$L_\epsilon(\lambda) \simeq L_\epsilon(\lambda_0) \otimes Fr^* L(\lambda_1)$$

Proof (i) Fin-dim irred U_ϵ modules -res to irred u_ϵ-modules whose highest weights are restricted

(ii) $M := L_\epsilon(\lambda_0) \otimes Fr^* L(\lambda_1)$.

As vector spaces, $Fr^* L(\lambda_1) \simeq \mathrm{Hom}_{u_\epsilon}(L_\epsilon(\lambda_0), M)$ (Schur's lemma)

$\mathrm{Hom}_{u_\epsilon}(L_\epsilon(\lambda_0), M) \subset \mathrm{Hom}_{\mathbb{C}}(L_\epsilon(\lambda_0), M) \in U_\epsilon^{\circ}$-mod.

$\mathrm{Hom}_{u_\epsilon}(L_\epsilon(\lambda_0), M) = \mathrm{Hom}_{\mathbb{C}}(L_\epsilon(\lambda_0), M)^{u_\epsilon}$

$= \{ f \in \mathrm{Hom}_{\mathbb{C}}(L_\epsilon(\lambda_0), M) \mid z(f) = \varepsilon(z) f \}$, ($\varepsilon$ is the counit map of u_ϵ).

$\Rightarrow \mathrm{Hom}_{\mathbb{C}}(L_\epsilon(\lambda_0), M)^{u_\epsilon}$ has the structure of $U_\epsilon^{\circ} / \{ x(z - \varepsilon(z)) \mid x \in U_\epsilon, z \in u_\epsilon \}$

$\simeq U(\mathfrak{g})$-module.

Moreover U_ϵ° is generated by $E_i, F_i, K_i^{\pm 1}$ & $E_i^{(l)}, F_i^{(l)}$, and brackets w/ all of them preserve the small quantum group (Lusztig Quantum group at roots of 1)

Here the $U(\mathfrak{g})$-action on $Fr^* L(\lambda_1)$ is the same as that on $L_\epsilon(\lambda_0)$.

(iii) Compare the highest weights.

$\square$

Notations

- $\mathbb{B} := \mathbb{Z}[v]_{\mathfrak{m}}$, $\mathfrak{m} := \langle v-1, p \rangle \triangleleft \mathbb{Z}[v]$ for some odd prime p.

We assume $p > 3$ if (a_{ij}) has a component of type G_2.

- $\mathbb{B}' := \mathrm{Frac}(\mathbb{B}) = \mathbb{Q}(v)$.
- $\Bbbk$ - residue field of $\mathbb{B}$ $\mathbb{F}_p$
- Γ: a $\mathbb{B}$-algebra
- $U_{\mathbb{B}}$: Lusztig's form
- $U_{\mathbb{B}}^{+}$ (resp. $U_{\mathbb{B}}^{-}$) : the $\mathbb{B}$-subalgebra generated $E_i^{(r)}$ (resp. $F_i^{(r)}$) $r \geq 0$.

- Specializations of $\mathbb{B}$.

Let $f: \mathbb{B} \to \Gamma$ be a specialization into a field.

(i) $\mathrm{char}(\Gamma) = p$

$$\begin{tikzcd} \mathbb{Z}[v, v^{-1}]_{(p, 1-v)} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma \\ \mathbb{F}_p[v, v^{-1}]_{(1-v)} \arrow[ur] & \end{tikzcd}$$

$f(v) = 1$

(ii) $\mathrm{char}(\Gamma) = 0$

Determine $f(v)$. It can be

- an element which is transcendental $/ \mathbb{Q}$

· algebraic / $\mathbb{Q}$. $\Rightarrow$ minimal polynomial F of $f(1)$ belong $(p, 1-v)$ i.e. $1-v$ divides F mod p.

In particular, if $f(1)$ is a root of 1, then it is a p^m-th root of 1 for some $m \in \mathbb{N}$. (All cyclotomic polynomial are divided by $1-v$ mod p).

§. Quantum Weyl Module

· **Def.** (Quantum Weyl Module)

For $\lambda \in \Lambda^+$, define the Q.W.M. to be

$$W_{\mathcal{B}}(\lambda) = U_{\mathcal{B}} \Big/ U_{\mathcal{B}} \left(\begin{array}{l} E_i^{(n_i)} \mid i = 1, \dots, n \\ z - \chi_\lambda(z) \mid z \in U_{\mathcal{B}}^0 \\ F_i^{(m_i)} \mid i = 1, \dots, n, \ n_i > (\lambda . \alpha_i^\vee) \end{array} \right)$$

· **Lemma** (i) $W_{\mathcal{B}}(\lambda)$ is a highest w.t. module w $\mid$ $W_{\mathcal{B}}(\lambda)_\mu \neq 0 \Rightarrow \mu \leq \lambda$.

(ii) $W_{\mathcal{B}}(\lambda)$ is integrable i.e. $\forall$ $w \in W_{\mathcal{B}}(\lambda)$, $\exists$ $r = r(w) \in \mathbb{N}$ s.t. $E_i^{(n)} \cdot w = F_i^{(n)} w = 0$, $\forall$ $n \geq r$, $i = 1 \dots n$. In particular, the set of weights of $W_{\mathcal{B}}(\lambda)$ is finite

(iii) $W_{\mathcal{B}}(\lambda)$ is f.g. / $\mathcal{B}$.

Proof (i) Observe $W_{\mathcal{B}}(\lambda) =$ quotient of $\Delta_{\mathcal{B}}(\lambda) := U_{\mathcal{B}} \otimes_{U_{\mathcal{B}}^{\geq 0}} \mathcal{B}_\lambda$

(ii) Follows from (i)

(iii) Follows from (i) and (ii)

· **Prop.** (i) The specialization $U_{\mathbb{B}'} = \mathbb{B}' \otimes_{\mathbb{B}} U_{\mathbb{B}}$ is the usual quantum group. The Weyl module $W_{\mathbb{B}'}(\lambda)$ is finite dimensional and irred.

(ii) We know that $U_{\mathbb{K}} = \mathrm{Dist}(G_{\mathbb{K}})\langle K_i \mid i=1,\dots,n\rangle / (K_i^2=1)$, and K_i's act by 1 and $W_{\mathbb{K}}(\lambda)$ becomes a $\mathrm{Dist}(G_{\mathbb{K}})$-module. In fact $W_{\mathbb{K}}(\lambda) = W'(\lambda)$, where $W'(\lambda)$ is defined in the previous talk, and its character is given by the W.C.F.

· **Coro.** $W_{\mathbb{B}}(\lambda)$ is free over $\mathbb{B}$.

· **Remarks** (i) The Weyl module $W_{\mathbb{C},\epsilon}(\lambda)$ has the desired universal prop. when ϵ is a p^m-th root of 1.

(ii) The method which allows us to go from $\mathrm{Dist}(G_{\mathbb{K}})$ to $U_{\mathbb{B}}$ can pushed further (see APW). For example, one can

- define dual Weyl modules using coinduction functor
- prove an analogue of the Kempf's vanishing Thm.
- prove that there are no higher Ext's between Weyl modules and dual Weyl modules.