Category $\mathcal{O}$ and its basic properties

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1 Definitions

Let $\mathfrak{g}$ denote a semisimple Lie algebra over $\mathbb{C}$ with fixed Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Define $\mathfrak{n} = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$. As we know $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. Also there is a partial ordering on the set of weights defined as follows: $\lambda \geq \mu \Leftrightarrow \lambda - \mu$ is equal to a $\mathbb{Z}^+$-linear combination of positive roots.

We start with the definition of BGG category $\mathcal{O}$:

Definition 1. $\mathcal{O}$ is the full subcategory in the category $U(\mathfrak{g})$-mod such that its objects satisfy the following properties:

1. The action of $\mathfrak{h}$ is semisimple.
2. The action of $\mathfrak{n}$ is locally finite.
3. The module is finitely generated over $U(\mathfrak{g})$.

Note that 1 implies the weight decomposition $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$, where $M_\lambda = \{v \in M | h \cdot v = \lambda(h) \cdot v \ \forall h \in \mathfrak{h}\}$.

Also we will use notions of highest weight and singular vectors:

Definition 2. A weight vector $v \in M$ of weight λ is called highest weight, if for any other weight vector $w \in M$ of weight η, $\eta \not> \lambda$.

Definition 3. A weight vector $v \in M$ is called singular, if for any $e \in \mathfrak{n}$, $e \cdot v = 0$.

Note that obviously any highest weight vector is also singular.

2 Basic properties of $\mathcal{O}$

Theorem 1. *$\mathcal{O}$ is abelian and if $M \in \mathcal{O}$, then:*

1. *M is finitely generated over $U(\mathfrak{h})$ by weight vectors.*
2. *M is finitely generated over $U(\mathfrak{n}_-)$ by weight vectors.*
3. *any weight space M_λ is finite dimensional.*
4. *M has a highest weight vector.*
5. *For every weight λ appearing in M the number of other weights μ appearing in M such that $\mu > \lambda$ is finite.*
6. *$\mathfrak{n}$ acts locally nilpotently.*
7. *$M \otimes N \in \mathcal{O}$ for every finite dimensional $\mathfrak{g}$-module N.*

Proof. $\mathfrak{g}$-mod is an abelian category and it is trivial to check that $\mathcal{O}$ is closed under submodules, quotients and direct sums.

1. By definition M is generated by a finite number of finite sums of weight vectors, hence by a finite number of weight vectors.

2. Since the action of $\mathfrak{n}$ is locally finite and the action of $\mathfrak{h}$ is diagonalizable it follows that the action of $\mathfrak{b}$ is also locally finite. Hence $U(\mathfrak{b})v$ is finite dimensional. Now from the fact that $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{b})$ it follows that $\bigoplus U(\mathfrak{b})v_i$, where v_i are generators from previous statement, generate M over $U(\mathfrak{n}_-)$.

3. From the previous statement it follows that vectors in M_λ can be obtained by the action of finitely many elements of $U(\mathfrak{n}_-)$ on the finite number of weight generators, hence, M_λ is finite dimensional.

4. Take any vector of maximal weight from the family of $U(\mathfrak{n}_-)$ weight generators.

5. From the second statement it follows that every weight μ appearing in M should be smaller or equal to at least one of the weights λ_i corresponding to $U(\mathfrak{n}_-)$ weight generators of M. Let's fix some weight η. If $\lambda_i \ngeq \eta$, then every weight $\mu \leq \lambda$ is also not bigger than η. If $\lambda \geq \eta$, then the number of weights such that $\lambda \geq \mu \geq \eta$ is obviously finite. Since the number of weights λ_i is finite, the statement follows.

6. For every weight vector $v \in M_\lambda$ there is a finite number of weights $P_v = \{\mu | M_\mu \neq 0, \mu > \lambda\}$. Consider $l_v = \sup_{\mu \in P} l(\mu - \lambda)$. Every word in $U(\mathfrak{n})$ with length bigger than l_v will act by zero on v. For any vector v' consider its weight decomposition $v' = \sum v_i$ and take maximal l_{v_i}.

7. The first axiom of category $\mathcal{O}$ is satisfied since every finite dimensional module is $\mathfrak{h}$-semisimple. Every vector in $M \otimes N$ is a finite sum of tensor products of weight vectors. $U(\mathfrak{n})v \otimes w$ is contained in $U(\mathfrak{n})v \otimes U(\mathfrak{n})w$ and is finite dimensional. Hence for a finite sum of such vectors their $U(\mathfrak{n})$-span is contained in the union of finite dimensional spaces and is too finite dimensional. To conclude we will show that if v_i generate M and w_i serve is a basis of N, $v_i \otimes w_j$ would generate $M \otimes N$. Let us prove this by induction. Suppose we have proved that these vectors generate every vector of the form $x \cdot v_i \otimes w$, where $x \in U(\mathfrak{g})$ is a word of the length lesser then n and $w \in N$. For any element $s \in \mathfrak{g}$, $s \cdot (x \cdot v_i \otimes w) = sx \cdot v_i \otimes w + x \cdot v \otimes s \cdot w$. But $x \cdot v \otimes s \cdot w \in U(\mathfrak{g}) \cdot \mathrm{Span}(\{v_i \otimes w_j\})$ by assumption. Hence we have proved that we can generate any vector of the form above with the length of the word lesser than $n + 1$.

□

Also we will need the notion of the character in our talk.

Definition 4. For $M \in \mathcal{O}$ denote by chM a formal sum $chM := \sum_{\lambda \in \mathfrak{h}^*} M_\lambda e^\lambda$.

Since all weight spaces are finite dimensional and lie in the union of $\lambda_i + \Lambda^+$ for finite number of λ_+ this formal series is well defined. For some objects this series converges. For example using $\Delta(\lambda) \simeq S(\mathfrak{n}_-)$ one can easily obtain the formula for the character of Verma module:

$$ch\Delta(\lambda) = e^\lambda \sum (S(\mathfrak{n}_-))_\mu e^\mu = e^\lambda \prod_{\alpha > 0} \frac{1}{1 - e^\alpha}$$

Later we will need the following lemma.

Lemma 1. *$Ext^1_{\mathcal{O}}(\Delta(\lambda), \Delta(\mu)) = 0$ for $\mu \ngtr \lambda$.*

Proof. Proof is an exercise. □

3 Simple objects

Suppose $L \in \mathcal{O}$ is simple. By Theorem 1.4 there is a highest weight vector $v \in L$, therefore there exists a homomorphism from Verma module $\phi : \Delta(\lambda) \to L$. The image of ϕ is a non-zero submodule of L, thus it equals to L and $L \simeq \Delta(\lambda)/Ker\phi$. We have showed that every simple object is isomorphic to a quotient of Verma module. Now let's study submodules of Verma module.

Proposition 1. *$\Delta(\lambda)$ has a unique proper maximal submodule $N(\lambda)$.*

Proof. Any proper submodule of $\Delta(\lambda)$ intersects $\Delta(\lambda)_\lambda$ by zero. Therefore the sum of any family of proper submodules is a proper submodule. Now we can take the union of all proper submodules to be a maximal proper submodule. □

From this it follows:

Theorem 2. *Every simple object L is isomorphic to $L(\lambda) \simeq \Delta(\lambda)/N(\lambda)$ and $L(\lambda) \not\simeq L(\mu)$ if $\lambda \neq \mu$.*

Proof. The first part of the theorem is obvious from Proposition 1 and the discussion above. The second statement follows from the fact that every vector in $L(\lambda)$ has a weight less or equal to λ and therefore if $L(\lambda) \simeq L(\mu)$ we have $\mu \leq \lambda$, but on the other hand $\mu \geq \lambda$, thus $\lambda = \mu$. □

4 Finite length

Theorem 3. *Every $M \in \mathcal{O}$ has finite length.*

Proof. As we know from Theorem 1.3 every module $M \in \mathcal{O}$ is finitely generated over $U(\mathfrak{n}_-)$ by a finite set of weight vectors, let V denote this set of generators. Let's show that this leads to M being a finite successive extension of quotients of Verma modules. Indeed, take a highest weight vector v_1 from V. The vector v_1 generates a submodule N_1 isomorphic to a quotient of Verma module. Take a quotient $M_1 = M/N_1$. This module is $U(\mathfrak{n}_-)$-generated by a smaller set $V_1 = V\backslash\{v\}$. Repeat this procedure a finite number of times to get the desired result.

Now to prove that every module in $\mathcal{O}$ has finite length it suffices to show that Verma modules have finite length. In order to do so we will need the action of the center $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ introduced in the previous talk.

It follows that only $L(\mu)$ with $\mu = w \cdot \lambda$ can appear in the composition series of $\delta(\mu)$. But $\dim L(\mu)_\mu = 1$ and $\dim \Delta(\lambda)_{w\cdot\lambda} < \infty$ for every w (Thm. 1.3). Thus there is a finite number of simple modules in the composition series of $\Delta(\lambda)$. □

5 Infinitesimal blocks

For $M \in \mathcal{O}$ and central character χ let us set $M^\chi = \{v \in M |$ for every $z \in Z(\mathfrak{g}) \Rightarrow (z - \chi(z))^n v = 0$ for some $n\}$, it is obviously a submodule of M. As we know from section 4 every module is a finite successive extension of quotients of Verma modules. But the center acts by a single character on Verma module, therefore there is only a finite number of characters χ such that M^χ is non zero. Also from this it follows that $M = \sum_\chi M^\chi$. Now we can define infinitesimal blocks:

Definition 5. The full subcategory $\mathcal{O}_\chi \subset \mathcal{O}$ is called infinitesimal block. It consist of modules such that $Z(\mathfrak{g})$ acts on them with a single eigenvalue equal to χ.

Obviously $\mathcal{O} = \sum_\chi \mathcal{O}_\chi$. By χ_λ we denote the central character by which $Z(\mathfrak{g})$ acts on $\Delta(\lambda)$. From the Harish-Chandra isomorphism it easily follows that $\mathcal{O}_{\chi_\lambda} = \mathcal{O}_{\chi_{w\cdot\lambda}}$ ($w \cdot \lambda$ is the twisted action of the Weyl group) and that simple objects of O_{χ_λ} are precisely $L(w\cdot\lambda)$.

6 Dominance and antidominance

Many properties of categories $\mathcal{O}_{\chi_\lambda}$ follow from the properties of the partial order restricted to the twisted Weyl orbit of λ. For example if we know that the set $\{w \cdot \lambda\}$ splits into subsets consisting of incomparable weights it follows that $\mathcal{O}_{\chi_\lambda}$ further decomposes into the sum of smaller subcategories.

It will be convenient for us to give a few definitions.

Definition 6. A weight λ is called dominant if it is maximal in its Weyl orbit. It is called antidominant if it is minimal in its Weyl orbit.

For example it's obvious that any weight lying in the positive Weyl chamber is dominant. Note that in terms of previous talk λ is dominant if it is maximal with respect to $\preccurlyeq$ and antidominant if it is minimal with respect to the same order.

This definition can be used to describe a class of irreducible Verma modules. Indeed, observe that if λ is antidominant $\Delta(\lambda)$ is irreducible since the only highest weight vectors its submodule can have is of weight λ.

Also there is an explicit description of dominant and antidominant weights which we will not prove or use.

Proposition 2. *A weight λ is dominant if and only if for any positive root α: $< \lambda + \rho, \alpha^\vee > \neq -1, -2, \ldots$. It is antidominant if and only if $< \lambda + \rho, \alpha^\vee > \neq 1, 2, \ldots$.*

Proof. Chapters 3.4-3.5 in Humphreys book on category O. □

7 $\mathcal{K}_0(\mathcal{O}_{\chi_\lambda})$

As usual we know that the classes of simple objects $[L(w\cdot\lambda)]$ freely generate $\mathcal{K}_0(\mathcal{O}_{\chi_\lambda})$. But in this particular case we also have another basis consisting of classes of Verma modules. Indeed, consider $\Delta(\mu) \in \mathcal{O}_{\chi_\lambda}$ and its class in $\mathcal{K}_0$. As we already know $\Delta(\mu)$ has $L(\mu)$ as a unique simple quotient. Because $(\Delta(\mu))_\mu \simeq \mathbb{C}$ and all over weights appearing in $\Delta(\mu)$ are smaller, we conclude that:

$$[\Delta(\mu)] = [L(\mu)] + \sum_{\eta<\mu} a_\eta [L(\eta)] \ ,$$

for some integers a_η. From this we see that classes of Verma modules are expressed in terms of classes of simple modules in an upper triangular way. Therefore we can inverse this expression to obtain an upper triangular expression of $[L(\mu)]$ in terms of $[\Delta(\eta)]$, and therefore $[\Delta(\eta)]$ form a basis of $\mathcal{K}_0(\mathcal{O}_{\chi_\lambda})$.

The other useful property of this category is that the classes in $\mathcal{K}_0$ are separated by characters. In other words the expression of $[M]$ in terms of some fixed basis of $\mathcal{K}_0$ can be obtained from the $ch(M)$. To prove this it suffices to show that we can do this for some fixed basis. Pick a basis consisting of $[\Delta(\eta)]$. Obviously $ch(\Delta(\eta)) = e^\eta \prod_{\alpha>0}(1 - e^{-\alpha})^{-1}$. Since classes of Verma modules form a basis, we have $[M] = \sum_\eta a_\eta [\Delta(\eta)] \Rightarrow ch(M) = \sum_\eta a_\eta ch(\Delta(\eta))$. Therefore $ch(M) \prod_{\alpha>0}(1 - e^{-\alpha}) = \sum_\eta a_\eta e^\eta$ and we are done.

8 Weyl's formula for character

Let's fix a dominant weight λ (so that $L(\lambda)$ is finite dimensional). From section 7 it follows that:

$$chL(\lambda) = \sum_{w\in W} n_w \cdot ch\Delta(w \cdot \lambda) \ ,$$

with $n_1 = 1$. As we know $ch\Delta(w \cdot \lambda) = e^{w(\lambda+\rho)-\rho}\prod_{\alpha>0}(1-e^{-\alpha})^{-1}$, hence:

$$chL(\lambda) = \sum_{w\in W} n_w e^{w(\lambda+\rho)} e^{-\rho} \prod_{\alpha>0} \frac{1}{1-e^{-\alpha}} \ .$$

As we know $L(\lambda)$ is W-invariant and one can easily see that $w(e^{-\rho}\prod_{\alpha>0}\frac{1}{1-e^{-\alpha}}) = w(\prod_{\alpha>0}\frac{1}{e^{\alpha/2}-e^{-\alpha/2}}) = (-1)^{l(w)}e^{-\rho}\prod_{\alpha>0}\frac{1}{1-e^{-\alpha}}$. Therefore $n_w = (-1)^{l(w)}n_1$ and we have the Weyl formula for characters:

$$chL(\lambda) = \sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho} \prod_{\alpha>0} \frac{1}{1-e^{-\alpha}} \ .$$

9 Duality

One can easily check that there exists an involutive automorphism τ of $\mathfrak{g}$ that acts on generators by the following formula: $\tau(h_i) = -h_i \ , \ \tau(e_i) = f_i \ , \ \tau(e_i) = f_i$. We will use this involution to define the notion of duality in $\mathcal{O}$.

Definition 7. For $M \in \mathcal{O}$ define the dual module $M^\vee$ to be equal to $\bigoplus_\lambda M_\lambda^*$ as a vector space and define a $\mathfrak{g}$-action on it by the following formula:

$$(x \cdot f)(v) = f(-\tau(x) \cdot v) \ ,$$

for $x \in \mathfrak{g} \ , \ v \in M \ , \ f \in M^\vee$.

Now we need to prove that $M^\vee \in \mathcal{O}$. The semisimplisity of the $\mathfrak{h}$-action is obvious. The local finiteness of the $\mathfrak{n}$-action easily follows from the fact that for $f \in M^\vee_\lambda$, $e_\alpha f \in M^\vee_{\lambda+\alpha}$. From this it already follows that any finitely generated submodule of $M^\vee$ is in $\mathcal{O}$. The only thing left to check is that $M^\vee \in \mathcal{O}$.

In order to do so, observe that $\bullet^\vee$ sends exact sequences to exact sequences and $M^{\vee\vee} \simeq M$, therefore if M is simple, then $M^\vee$ is simple too and therefore belongs to $\mathcal{O}$. Hence $L(\lambda)^\vee \simeq L(\lambda)$ because their highest weights coincide. Now since M has finite length, $M^\vee$ also has finite length with the same composition factors, therefore it is finitely generated and belongs to $\mathcal{O}$.

Let's discuss the properties of this duality functor. It is clear that $\nabla(\lambda) := \Delta(\lambda)^\vee$ has a unique simple submodule $L(\lambda)$. The other useful properties are as follows:

Proposition 3. *1.* $\dim Hom_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda\mu}$

2. $Ext^1_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu)) = 0$

Proof. 1. Since any homomorphism of Verma module is uniquely determined by the image of the highest weight vector, we see that $Hom_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu))$ is zero for all λ except $\lambda \le \mu$. The image of a nonzero homomorphism is a submodule of $\nabla(\mu)$, therefore it contains $L(\mu)$. Hence $\mu \le \lambda$ and Hom is nonzero only for $\lambda = \mu$. In this case the homomorphism is defined by $\Delta(\lambda) \to L(\lambda) \to \nabla(\lambda)$ and is unique up to a scalar since $\dim(\nabla(\lambda))_\lambda = 1$.

2. Consider a short exact sequence:

$$0 \to \nabla(\mu) \to M \to \Delta(\lambda) \to 0 \ .$$

If μ is not greater then λ, then the preimage of a highest weight vector is a highest weight vector and the sequence splits. If $\mu > \lambda$ take the dual of the sequence to get:

$$0 \to \nabla(\lambda) \to M^\vee \to \Delta(\mu) \to 0 \ .$$

The new sequence splits and therefore the original sequence also splits.

$\square$

10 Standard filtration

As we already know each $M \in \mathcal{O}$ admits a filtration with successive quotients being isomorphic to quotients of Verma modules. Now we define a stronger notion.

Definition 8. We say that $M \in \mathcal{O}$ is standardly filtered if there is a chain of submodules $0 = F^0M \subset F^1M \subset F^2M \subset \cdots \subset F^nM = M$ such that each $F^{i+1}M/F^iM$ is isomorphic to a Verma module. Moreover, if M admits a standard filtration let $(M : \Delta(\lambda))$ denote the multiplicity with which $\Delta(\lambda)$ appears in the filtration.

The notion of multiplicity is well-defined, because $(M : \Delta(\lambda)$ can be obtained from the character of the module, if we know that a standard filtration exists. Now let's consider properties of the standard filtration.

Proposition 4. *Let $M,\ N \in \mathcal{O}$ have standard filtrations.*

1. *Their sum $M \oplus N$ also has a standard filtration and the multiplicity index is linear.*
2. *If λ is a maximal weight for M, then M has a submodule isomorphic to $\Delta(\lambda)$ and the quotient $M/\Delta(\lambda)$ has a standard filtration.*
3. *If $M = M_1 \oplus M_2$, then the summands also have standard filtrations.*
4. $(M : \Delta(\lambda)) = \dim Hom_{\mathcal{O}}(M, \nabla(\lambda))$

Note that the fourth statement of the proposition also proves that the multiplicity is independent of the choice of a filtration.

Proof.

1. The construction of a filtration for the sum is straightforward and linearity easily follows.
2. Since λ is a maximal weight, there is a highest weight vector in M of this weight and a homomorphism $\phi : \Delta(\lambda) \to M$. Let's pick the smallest i such that the image of ϕ is contained in F^iM. For this i the homomorphism $\psi : \Delta(\lambda) \to F^iM/F^{i-1}M \simeq \Delta(\mu)$ is non-zero. Now by maximality of λ we know that $\mu = \lambda$ and ψ is isomorphism and thus $\Delta(\lambda)$ is a submodule of M and $F^iM = \Delta(\lambda) \oplus F^{i-1}M$. Therefore we have a short exact sequence:
$$0 \to F^{i-1}M \to M/\Delta(\lambda) \to M/F^iM \to 0\ ,$$
which leads to a standard filtration on $M/\Delta(\lambda)$ of a smaller length.
3. Proceed by induction on the filtration length of M. If M is a Verma, it is indecomposable and there is nothing to prove. Otherwise, consider a maximal weight λ of M. Wlog assume $(M_1)_\lambda \neq 0$, then there is a homomorphism $\Delta(\lambda) \to M_1$. By the previous part $\Delta(\lambda)$ is a submodule of M and since the highest weight vector of $\Delta(\lambda)$ belongs to M_1 it is also a submodule of M_1. Also from the previous discussion we know that $M/\Delta(\lambda) \simeq (M_1/\Delta(\lambda)) \oplus M_2$ has a standard filtration. By induction M_2 and $M_1/\Delta(\lambda)$ have standard filtrations and therefore M_1 and M_2 also have a standard filtrations.
4. Proof follows from the fact that $Ext^1_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu)) = 0$. Completing the proof is an exercise.

□

In the next lecture the following fact about standard filtration will be proven.

Proposition 5. *Let L be a finite dimensional module, then module $\Delta(\lambda) \otimes L$ admits a standard filtration with multiplicities $(\Delta(\lambda) \otimes L : \Delta(\eta)) = \dim L_{\eta-\lambda}$.*

Also note that standard filtration is canonical in some sense. More precisely:

Proposition 6. *Let M be a standardly filtered module. Consider a set of weights B consisting of λ such that $(M : \Delta(\lambda)) \neq 0$. Order elements of this set $B = \{\lambda_1, \ldots, \lambda_k\}$ in such a way that $\lambda_i \geq \lambda_j \Rightarrow i \leqslant j$. Let $n_i := (M : \Delta(\lambda_i))$. Then there is a unique filtration $F^i M$ of length k such that $F^i M/F^{i-1}M \simeq \Delta(\lambda_i)^{\oplus n_i}$.*

Proof. Induction on the length of the filtration. From Proposition 4.2 it follows that since λ_1 is maximal $\Delta(\lambda_1)^{\oplus n_1}$ is a submodule of M. Fix $F^1M = \Delta(\lambda_1)^{\oplus n_1}$. Consider $\pi : M \to M/F^1M = M'$. The module M' has a standard filtration of desired form by hypothesis. Take $F^iM = \pi^{-1}(F^{i-1}M')$. □

11 Projective objects

Let's start by constructing some projective objects in $\mathcal{O}_\chi$.

Proposition 7. 1. *If λ is dominant, then $\Delta(\lambda)$ is projective.*

2. *If L is a finite dimensional module and P is projective, then $P \otimes L$ is projective.*

Proof.
1. Since λ is dominant, any weight vector $v \in M$ of such weight is singular. Hence $Hom_{\mathcal{O}}(\Delta(\lambda), M) = M_\lambda$ and the functor $Hom_{\mathcal{O}}(\Delta(\lambda), \bullet)$ is exact.

2. $Hom_{\mathcal{O}}(P \otimes L, M) = Hom_{\mathcal{O}}(P, L^* \otimes M)$ thus $Hom_{\mathcal{O}}(P \otimes L, \cdot)$ is the composition of two exact functors, hence exact.

□

Using these projective objects we can construct a projective object covering $L(\lambda)$ in the following way. For a sufficiently large n, $\lambda + n\rho$ is dominant. Take such n and consider $L(\lambda) \otimes L(n\rho)$. It has a highest weight vector of weight $\lambda + n\rho$ and hence it admits nonzero homomorphism $\Delta(\lambda + n\rho) \to L(n\rho) \otimes L(\lambda)$. The module $L(n\rho)$ is finite dimensional, thus the dual is also finite dimensional and we have: $\Delta(\lambda + n\rho) \otimes L(n\rho)^* \to L(\lambda)$. The homomorphism is non-zero, therefore it surjects and we are done.

Now using the induction on the length of the object one can easily prove that $\mathcal{O}$ has enough projectives. Indeed, consider $0 \to L(\lambda) \to N \to M \to 0$, by the induction hypothesis there exist projective objects P, P' and epimorphisms $P \to M$, $P' \to L(\lambda)$. By the defining property of projective objects we have $P \to N$ and it is clear that $P \oplus P' \to N$ is surjective.

Lemma 2. *Every $L(\lambda)$ admits an indecomposable projective object which surjects onto $L(\lambda)$. $P(\lambda)$ is unique up to isomorphism.*

Proof. Existence: As we already know there is a projective object P covering $L(\lambda)$. Since P has finite length it can be decomposed as a sum of indecomposable projectives $P = \oplus P_i$. $Hom_{\mathcal{O}}(P, L(\lambda)) = \oplus Hom_{\mathcal{O}}(P_i, L(\lambda))$, therefore one of P_i surjects onto $L(\lambda)$.

Uniqueness:Let P be another indecomposable projective covering $L(\lambda)$. By the projectivity of P there is a map $\psi : P \to P(\lambda)$ which is surjective since $P(\lambda)$ is a projective cover. By projectivity of $P(\lambda)$ there is a map $\psi' : P(\lambda) \to P$ such that $\psi\psi' = 1$, hence $P(\lambda)$ is a direct summand of P, but since P is indecomposable $P \simeq P(\lambda)$. □

Note that by the projective property we have epimorphisms $P(\lambda) \to \Delta(\lambda) \to L(\lambda)$. Also it easily follows that $Hom_{\mathcal{O}}(P(\lambda), L(\mu)) = \delta_{\lambda\mu}\mathbb{C}$, and hence these projective objects are non-isomorphic.

Now let's prove a proposition which we will need later on.

Proposition 8. $\dim Hom_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$, *where by* $[M : L(\lambda)]$ *we denote the multiplicity of* $L(\lambda)$ *in the composition series.*

Proof. Proof is by induction on length. Completing the proof is an exercise. $\square$

To finish this section we shall establish a standard filtration on projective objects. As we know $L(\lambda)$ admits a projective cover by $\Delta(\lambda + n\rho) \otimes L(n\rho)$. From Proposition 5 we know that this object has a standard filtration. $P(\lambda)$ is contained in this projective object as a direct summand, hence it has a standard filtration too.

Now we will suppose that $P(\lambda)$ has standard filtration and prove an important fact about this filtration (this fact also follows from Proposition 5).

Proposition 9. $(P(\lambda) : \Delta(\lambda)) = 1$ *and* $(P(\lambda) : \Delta(\mu)) = 0$ *for* $\mu \not> \lambda$.

Proof. As we know there is an epimorphism $P(\lambda) \to \Delta(\lambda)$, therefore $P(\lambda)$ admits a standard filtration with $\Delta(\lambda)$ on top. Suppose there are other Verma modules in the filtration with highest weight $\eta \not> \lambda$. Take such module with a minimal weight μ. We know that there is an i such that $F^i P(\lambda)/F^{i-1}P(\lambda) \simeq \Delta(\mu)$. Also we know that $F^{i+1}P(\lambda)/F^i P(\lambda) \simeq \Delta(\eta)$ with $\mu \not> \eta$. We have a short exact sequence $0 \to \Delta(\mu) \to F^{i+1}P(\lambda)/F^{i-1}P(\lambda) \to \Delta(\eta) \to 0$. From Lemma 1 we know that this sequence splits. Therefore we can change the filtration so that $\Delta(\mu)$ is higher in the filtration. By repeating this procedure we obtain a filtration with $\Delta(\mu)$ on top. Hence we get a epimorphism $P(\lambda) \to L(\mu)$ and its kernel surjects onto $L(\lambda)$, which leads to contradiction. $\square$

12 Highest weight structure

Definition 9. Consider an abelian category $\mathcal{C}$ which has a finite number of simple objects, enough projectives and every object has finite length (equivalently $\mathcal{C} \simeq A - mod$, where A is a finite dimensional associative algebra). The highest weight structure on such a category, is a partial order $\geq$ on $Irr(\mathcal{C})$ and the set of standard objects Δ_L , $L \in Irr(\mathcal{C})$ such that:

1. $Hom_{\mathcal{C}}(\Delta_L, \Delta_{L'}) \neq 0 \Rightarrow L' \geq L$ and $End_{\mathcal{C}}(\Delta_L) = \mathbb{C}$.
2. The projective cover P_L of L admits an epimorphism onto Δ_L and $Ker(P_L \to \Delta_L)$ admits a filtration by $\Delta_{L'}$ with $L' > L$.

From the previous discussion it is obvious that $\mathcal{O}_\chi$ has a highest weight structure with $\geq$ being a standard order on $\mathfrak{h}^*$ and $\Delta_{L(\lambda)} = \Delta(\lambda)$.

13 BGG reciprocity

Theorem 4. $(P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)] = [\nabla(\mu) : L(\lambda)]$

Proof. The second equality is obvious since M and $M^\vee$ have the same simple modules in their composition series. From Proposition 8 we know that $\dim Hom_{\mathcal{O}}(P(\lambda), \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$. And from part 4 of Proposition 4 we know that $(P(\lambda) : \Delta(\mu)) = \dim Hom_{\mathcal{O}}(P(\lambda), \nabla(\mu))$. This proves the theorem. $\square$