Recall that in [K][Theorem 3.3.1] we proved

**Theorem 0.1.** There is a canonical isomorphism

\[ \mathfrak{z}(\hat{\mathfrak{sl}}_2)_x \cong \mathbb{C}[\text{Proj}(D_x)], \]

where \( \text{Proj}(D_x) \) is the space of projective connections on \( D_x := \text{Spec}(\mathcal{O}_x). \)

In this note we will show that Theorem 0.1 implies

**Theorem 0.2.** There is a canonical isomorphism

\[ \mathcal{Z}(\tilde{U}_\kappa(\mathfrak{g})) := \mathcal{Z}(\hat{\mathfrak{sl}}_2)_x \cong \mathbb{C}[\text{Proj}(\tilde{D}_x)], \]

where \( \text{Proj}(\tilde{D}_x) \) is the space of projective connections on \( \tilde{D}_x := \text{Spec}(\mathcal{K}_x). \)

We will also generalize the statements in the following way. Let \( G \) be a simply-connected semi-simple algebraic group with the root system \( \Delta. \) Let \( \hat{\Delta} \) be the dual root system.

**Definition 0.1.** Langlands dual group \( \hat{G} \) is the adjoint group with root system \( \hat{\Delta}. \)

The goal of this seminar is to prove

**Theorem 0.3.** There is a canonical isomorphism

\[ \mathfrak{z}(\hat{\mathfrak{g}})_x \cong \mathbb{C}[^{\hat{G}}\text{Op}(D_x)], \]

where \( ^{\hat{G}}\text{Op}(D_x) \) is the space of opers, which will be analogs of projective connections for general \( \hat{G}. \)

**Theorem 0.4.** There is a canonical isomorphism

\[ \mathcal{Z}(\hat{\mathfrak{g}})_x \cong \mathbb{C}[^{\hat{G}}\text{Op}(D_x)]. \]

1. **Generalities on connections.**

Throughout this section let \( G \) be adjoint group. Let \( X \) be a smooth variety, \( \mathcal{P} \) be a principal \( G \)-bundle. Abusing notation we will also denote the total space of this principal \( G \)-bundle by \( \mathcal{P}, \) and projection to \( X \) by \( f. \)

We have a \( G \)-equivariant short exact sequence

(1.1) \[ 0 \to f^*\Omega^1_X \to \Omega^1_{\mathcal{P}} \to \mathcal{O}_\mathcal{P} \otimes \mathfrak{g}^* \to 0. \]

**Definition 1.1.** A connection \( \nabla \) on \( \mathcal{P} \) is a \( G \)-equivariant section of (1.1). Or, equivalently, a section \( \mathcal{P} \times_G \mathfrak{g}^* \to (f, \Omega^1_{\mathcal{P}})^G \) of

(1.2) \[ 0 \to \Omega^1_X \to (f_*\Omega^1_{\mathcal{P}})^G \to \mathcal{P} \times_G \mathfrak{g}^* \to 0, \]

where \( \mathcal{P} \times_G \mathfrak{g}^* \) is, by definition, the quotient of \( \mathcal{P} \times \mathfrak{g}^* \) by the diagonal action.

Given a trivialization of \( \mathcal{P} \) we get another section. Namely, then we get \( \mathcal{P} \cong X \times G \) and therefore

\[ \Omega^1_{\mathcal{P}} \cong f^*\Omega^1_X \oplus (\mathcal{O}_\mathcal{P} \otimes \mathfrak{g}^*). \]

Taking the difference we obtain an element of \( \text{Hom}(\mathcal{P} \times_G \mathfrak{g}^*, \Omega^1_X) \cong \text{Hom}(\mathcal{O}_X, (\mathcal{P} \times_G \mathfrak{g}) \otimes \Omega^1_X). \)
Remark 1.2. Let $X$ be a smooth 1-dimensional variety. Choose a local coordinate $z$. Then $\mathcal{P}$ trivializes and the data of the connection is an element of $\text{Hom}(\mathcal{O}_X, \mathfrak{g} \otimes \Omega^1_X)$, and can be given by
\begin{equation}
\nabla = dz + A(z)dz,
\end{equation}
where $A$ is a function valued in $\mathfrak{g}$. In this formula, $d$ stands for the connection coming from the trivialization of $\mathcal{P}$.

Below we will abuse the notation and remove the $dz$ from the formula (1.3). We will write a connection in the form
\[ \partial_s + A(z). \]

Exercise 1.3. Under change of trivialization of $\mathcal{P}$ by a function $g : X \rightarrow G$ we have
\[ \partial_s + A(z) \mapsto \partial_s + gA(z)g^{-1} - (\partial_s g)g^{-1}. \]
These are called gauge transformations.

Remark 1.4. Giving a section $s$ of $\mathcal{P} \times_G (\text{PGL}_2/B)$ is equivalent to giving a $B$-reduction $\mathcal{P}_B$ of $\mathcal{P}$. Indeed, for $\mathcal{P}_B$ take a fiber product
\[ \begin{array}{ccc}
\mathcal{P}_B & \longrightarrow & \mathcal{P} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{P} \times_G (\text{PGL}_2/B).
\end{array} \]

Construction 1.5. Choose a local section $s'$ of $\mathcal{P}$ lifting $s$. This gives a trivialization of $\mathcal{P}$ and hence an element of $\text{Hom}(\mathcal{O}_X, \mathcal{P} \times_G \mathfrak{g} \otimes \Omega^1_X)$. Project to $\text{Hom}(\mathcal{O}_X, \mathcal{P}_B \times_B \mathfrak{g}/b \otimes \Omega^1_X)$. The result is called the derivative of $s$ along $\nabla$. Note that it does not depend on $s'$.

2. Oper for $\text{PGL}_2$.

Set $X = \text{Spec}(\mathbb{C}[[t]])$ or $\text{Spec}(\mathbb{C}((t)))$.

Notation 2.1. Denote $D := \text{Spec}(\mathbb{C}[[t]])$ and $\hat{D} := \text{Spec}(\mathbb{C}((t)))$.

Definition 2.2. A $\text{PGL}_2$-oper on $X$ is a principal $\text{PGL}_2$-bundle $\mathcal{F}$ over $X$ with a connection $\nabla$, plus a globally defined section $s$ of the associated $\mathbb{P}^1$-bundle $\mathcal{F} \times_{\text{PGL}_2} (\text{PGL}_2/B)$, which has a nowhere vanishing derivative along $\nabla$.

Remark 2.3. Note that although $\text{Spec}(\mathbb{C}[[t]])$ and $\text{Spec}(\mathbb{C}((t)))$ are not varieties, the constructions from previous sections still make sense.

For $D$ set $\Omega^1_D := \mathbb{C}[[t]]dt$, and note that all $G$-bundles on $D$ are trivial.

For $\hat{D}$ set $\Omega^1_{\hat{D}} := \mathbb{C}((t))dt$, and note by [S][III.2.3. Theorem 1’] that all $G$-bundles on $\hat{D}$ are trivial.

Remark 2.4. Definition 2.2 makes sense for a smooth curve as well, but we will specialize to $X$.

The condition that the derivative of $s$ along $\nabla$ is nowhere vanishing says that $\nabla$ does not preserve $\mathcal{F}_B$ at any point. More precisely:

Exercise 2.5. Let $(\mathcal{F}, \nabla, s)$ be a $\text{PGL}_2$-oper. Choose a lift of $s$ to $\mathcal{F}$ to get a trivialization of $\mathcal{F}$. Then
\[ \nabla_{\partial_t} = \partial_t + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \]
where $a(t) + d(t) = 0$. The derivative of $s$ along $\nabla$ is nowhere vanishing if and only if $c(t)$ is nowhere vanishing (i.e. invertible element of $\mathbb{C}[[t]]$ or $\mathbb{C}((t))$).
Lemma 2.6. Let \((\mathcal{F}, \nabla, s)\) be as above. Then \(\nabla\) can be brought to a form
\[
\partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}
\]
in a unique way. So we get that PGL\(_2\)-opers non-canonically form an (ind)-affine space in the sense of algebraic geometry.

Proof. Apply the gauge transformation by
\[
\begin{pmatrix} c(t) & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2
\]
and get a connection of the form
\[
\partial_t + \begin{pmatrix} a_1(t) & b_1(t) \\ 1 & d_1(t) \end{pmatrix}.
\]
Then apply the gauge transformation by
\[
\begin{pmatrix} 1 & -a_1(t) \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2.
\]
\(\Box\)

Remark 2.7. Another way to realize a PGL\(_2\)-oper \((\mathcal{F}, \nabla, s)\) is the following. Trivialize the PGL\(_2\)-bundle \(\mathcal{F}\) and choose a lift \(\tilde{\mathcal{F}}\) to an SL\(_2\)-torsor with connection \(\nabla\). The lift is unique up to tensoring with line bundles that square to \(\mathcal{O}_X\) with connection.

Section of the associated \(\mathbb{P}^1\)-bundle \(s\) gives a \(B\)-reduction, which gives rize to a line subbundle \(\mathcal{F}_1 \subset \tilde{\mathcal{F}}\).

The connection \(\nabla\) on \(\tilde{\mathcal{F}}\) corresponds to a map \(\mathcal{O}_X \to \tilde{\mathcal{F}} \otimes \Omega^1_X\), and thus to a map \(\tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \otimes \Omega^1_X\).

An oper condition means that for
\[
0 \to \mathcal{F}_1 \xrightarrow{\nabla} \tilde{\mathcal{F}} \otimes \Omega^1_X \to \tilde{\mathcal{F}}/\mathcal{F}_1 \otimes \Omega^1_X \to 0
\]
is an isomorphism.

This means that
\[
0 \to \mathcal{F}_1 \otimes \Omega^1_X \otimes \tilde{\mathcal{F}}/\mathcal{F}_1 \otimes \Omega^1_X \cong \mathcal{F}^2_1 \otimes \Omega^1_X \to 0,
\]
and therefore
\[
\mathcal{F}^2_1 \otimes \Omega^1_X \cong \det \tilde{\mathcal{F}} \otimes \Omega^1_X,
\]
i.e.
\[
\mathcal{F}^2_1 \cong \Omega^1_X.
\]

Choose \(\mathcal{F}_1\) for a square root of \(\Omega^1_X\). We get
\[
0 \to \Omega^1_X \to \tilde{\mathcal{F}} \to \Omega^{-\frac{1}{2}}_X \to 0.
\]

Proposition 2.8. There is a one-to-one correspondence
\[
\{\text{PGL}_2\text{-opers on } X\} \leftrightarrow \{\text{Projective connections on } X\}.
\]

Proof. Let us canonically construct a projective connection from a PGL\(_2\)-oper. Let \((\tilde{\mathcal{F}}, \nabla)\) be as in Remark 2.7. We have
\[
0 \to \Omega^1_X \xrightarrow{i} \tilde{\mathcal{F}} \xrightarrow{\nabla} \Omega^{-\frac{1}{2}}_X \to 0
\]
such that \(\pi \circ \nabla \circ i\) is the identity.

Let us construct a differential operator \(\rho : \Omega^1_X \xrightarrow{i} \tilde{\mathcal{F}} \otimes \Omega^1_X \xrightarrow{\pi} \Omega^2_X \to 0\),

such that \(\pi \circ \nabla \circ i\) is the identity.

Let us construct a differential operator \(s' : \Omega^1_X \xrightarrow{i} \tilde{\mathcal{F}} \otimes \Omega^1_X \xrightarrow{\pi} \Omega^2_X \to 0\),

such that \(\pi \circ \nabla \circ i\) is the identity.

Let us construct a differential operator \(\rho(s) = \nabla(s),\)
where $\tilde{s} := s' - i \circ \pi \circ \nabla(s')$.

Now choose $s$ such that $s^2 = (dt)^{-1}$ (the choice is unique up to a sign). Consider a basis $(s^{-1}, \tilde{s})$, and in this basis

$$\nabla \partial_i = \partial_i + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}.$$  

**Exercise 2.9.** Using $s$ and $s^3$ as trivializations of $\Omega^1_X$ and $\Omega^3_X$ respectively, show that $\rho = \partial_t^3 + v(t)$.

So we constructed a canonical map which is seen to be a bijection. \hfill \Box

3. OPERS FOR GENERAL $G$.

To define opers for general $G$ we need to formulate an analog of the non-vanishing derivative condition.

Let $G$ be an adjoint group. Choose $g = n_+ \oplus h \oplus n_+$. This gives $n_+ \subset b \subset g$ and

$$N = [B, B] \subset B \subset G \supset H.$$  

Let $f_i$ be the standard generators of $n_-$, let $e_i$ be corresponding standard generators of $n_+$. Set $n_{\alpha_i} = C e_i$, $n_{-\alpha_i} = C f_i$.

Let $[n, n]^\perp \subset g$ be the orthogonal complement of $[n, n]$ w.r.t. $\kappa_0$.

**Lemma 3.1.** We have

$$[n, n]^\perp / b \cong \oplus_{i=1}^{rk} n_{-\alpha_i}.$$  

**Construction 3.2.** Note that $B$ acts on $[n, n]^\perp / b$. There is a unique open $B$-orbit

$$\mathcal{O} \subset [n, n]^\perp / b \subset g / b$$  

consisting of vectors with non-zero projection on each $n_{-\alpha_i}$. This orbit is isomorphic to $B / N$, where we use that $G$ is adjoint.

**Remark 3.3.** Note that $\mathcal{O}$ only depends on the choice of $b$.

**Construction 3.4.** Recall that $X = \text{Spec} (\mathbb{C}[[t]])$ or $\text{Spec} (\mathbb{C}((t)))$. Let $\mathcal{F}$ be a $G$-torsor on $X$ with a connection $\nabla$ and $B$-reduction $\mathcal{F}_B$. Choose any flat connection $\nabla'$ on $\mathcal{F}$ preserving $\mathcal{F}_B$ (we can do it since $\mathcal{F}_B$ is trivial) and take $\nabla = \nabla'$. This gives a section of $\mathcal{F} \times_G g \otimes \Omega^1_X \cong \mathcal{F}_B \times g \otimes \Omega^1_X$. Now project this section to $\mathcal{F}_B \times g / b \otimes \Omega^1_X$, and note that the result does not depend on $\nabla'$. We call this element of $\text{Hom}(\mathcal{O}_X, \mathcal{F}_B \times g / b \otimes \Omega^1_X)$ the relative position of $\nabla$ and $\mathcal{F}_B$ and denote it by $\nabla / \mathcal{F}_B$.

**Definition 3.5.** We say that $\mathcal{F}_B$ is transversal to $\nabla$ if

$$\nabla / \mathcal{F}_B \in \text{Hom}(\mathcal{O}_X, \mathcal{F}_B \times \mathcal{O} \otimes \Omega^1_X) \subset \text{Hom}(\mathcal{O}_X, \mathcal{F}_B \times g / b \otimes \Omega^1_X).$$  

**Remark 3.6.** When $G = \text{PGL}_2$ this is exactly the non-vanishing condition we had before.

**Definition 3.7.** A $G$-oper on $X$ is a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal $G$-bundle, $\nabla$ is a connection on $\mathcal{F}$ such that $\mathcal{F}_B$ is transversal to $\nabla$.

**Remark 3.8.** Choose a trivialization of $\mathcal{F}_B$. Then the oper condition says that

$$(\ref{eq:oper_condition_1}) \quad \nabla \partial_t = \partial_t + \sum_{i=1}^{rk} \psi_i(t) f_i + v(t),$$

where $\psi_i$ are non-vanishing functions (i.e. invertible elements of $\mathbb{C}[[t]]$ or $\mathbb{C}((t))$), and $v(t)$ is a $b$-valued function.

**Definition 3.9.** Let $\tilde{\text{Op}}_G(X)$ be the space of opers of the form

$$(\ref{eq:oper_condition_2}) \quad \nabla \partial_t = \partial_t + \sum_{i=1}^{rk} f_i + v(t),$$

where $v(t)$ is a $b$-valued function.
Proof. Recall that $B$-orbit $\mathcal{O}$ is an $H$-torsor, so we can bring a connection of the form (3.1) to the form (3.2) by a unique element of $H_X$. \hfill \Box

We are now going to describe the (ind)-scheme of opers using the Kostant slice.

Notation 3.12. Let $p_{-1} := \sum_{i=1}^{h} f_i$, and let $(p_{-1}, 2\rho, p_1)$ be the principal $sl_2$-triple.

Note that $\ker \mathrm{ad} p_{-1} \subset \mathfrak{b}$, and $p_{-1} + \mathfrak{b}$ is stable under the action of $\mathcal{N}$. Recall that the following two proposition and their corollaries were discussed in [K]:

**Proposition 3.13.** (Kostant) The map

$$N \times S \to p_{-1} + \mathfrak{b}$$

given by the action is an isomorphism.

**Corollary 3.14.** $N_X \times S_X \cong (p_{-1} + \mathfrak{b})_X$.

**Proposition 3.15.** (Kostant) The composition of the embedding $S \hookrightarrow \mathfrak{g}$ and the quotient morphism $\mathfrak{g} \rightarrow \mathfrak{g}/G$ is an isomorphism.

**Corollary 3.16.** $S_X \rightarrow \mathfrak{g}_X//G_X$.

We now apply them in the study of opers.

**Proposition 3.17.** The morphism of schemes

$$N[[t]] \times \{\partial_t + S[[t]]\} \to \{\partial_t + (p_{-1} + \mathfrak{b})[[t]]\} = \widetilde{\mathrm{Op}}_G(D)$$

induced by the gauge transformation action is an isomorphism.

**Proof.** Let $A^1_{\lambda} = \text{Spec}([\lambda])$ and let $\text{AffSch}_{/A^1_{\lambda}}$ be the category of affine schemes over $A^1_{\lambda}$. Consider $\{\lambda \partial_t + S[[t]]\}, \{\lambda \partial_t + (p_{-1} + \mathfrak{b})[[t]]\} \in \text{AffSch}_{/A^1_{\lambda}}$.

By definition, the gauge action of $\mathcal{N}(t) \in N[[t]]$ on $\{\lambda \partial_t + S[[t]]\}$ is given by

$$n(t) \cdot (\lambda \partial_t + s(t)) = \lambda \partial_t + \Ad(n(t)) s(t) - \lambda(\partial_t n(t)) n(t)^{-1}.$$  (3.4)

Note that $N[[t]] \times \{\lambda \partial_t + S[[t]]\}$ and $\{\lambda \partial_t + (p_{-1} + \mathfrak{b})[[t]]\}$ admit a $G_m$-action such that the action map is equivariant. Indeed, for $z \in \mathbb{C}^\times$ we have

$$z \cdot (n(t), \lambda \partial_t + s(t)) := (z \Ad\rho(z) n(t), z \lambda \partial_t + z \Ad\rho(z) s(t)),
$$

$$z \cdot (\lambda \partial_t + \Ad(n(t)) s(t) - \lambda(\partial_t n(t)) n(t)^{-1}) = z \lambda \partial_t + z \Ad\rho(z) \Ad(n(t)) s(t) + z \lambda \Ad\rho(z) (\partial_t n(t)) n(t)^{-1}.$$  

The restriction of the action map to the fiber $\lambda = 1$ is the desired map (3.3). This implies that there is a natural filtration on $\mathbb{C}[N[[t]]] \otimes \mathbb{C}[[\partial_t + S[[t]]]]$ and $\mathbb{C}[[\partial_t + (p_{-1} + \mathfrak{b})[[t]]]]$ such that the pullback map

$$\mathbb{C}[[\partial_t + (p_{-1} + \mathfrak{b})[[t]]]] \to \mathbb{C}[N[[t]]] \otimes \mathbb{C}[[\partial_t + S[[t]]]]$$

is filtered. Note that the filtration is non-negative. However, the restriction of the action map to the fiber $\lambda = 0$ is the isomorphism from Corollary 3.14. Thus the associated graded of (3.5) is an isomorphism, and hence (3.5) is also an isomorphism. \hfill \Box

**Corollary 3.18.** We have

$$\mathrm{Op}_G(D) \cong \{\partial_t + S[[t]]\}.$$

Moreover, we get that

$$\text{gr } \mathbb{C}[\mathrm{Op}_G(D)] \cong \mathbb{C}[S[[t]]] \cong \mathbb{C}[\mathfrak{g}[[t]]]^G[[t]],$$

and hence $\mathbb{C}[\mathrm{Op}_G(D)] \cong \mathbb{C}[\mathfrak{g}[[t]]]^G[[t]]$.  


We also have an analogous statement for loop spaces defined in Example 4.3:

**Proposition 3.19.** The morphism of ind-schemes

\[(3.6) \quad N((t)) \times \{ \partial_t + S((t)) \} \to \{ \partial_t + (p_{-1} + b)((t)) \} = \mathcal{O}_{p_G}(\hat{D})\]

induced by the gauge transformation action is an isomorphism.

**Corollary 3.20.** We have an isomorphism of ind-schemes

\[\mathcal{O}_{p_G}(\hat{D}) \cong \{ \partial_t + S((t)) \} \]

Finally, to make sense of the statement of Theorem 0.4 we need define algebras of functions on ind-schemes \(\mathcal{O}_{p_G}(\hat{D})\) in our case). This is addressed in Definition 4.8.


**Definition 4.1.** An ind-scheme is a functor \(\text{AffSch}^{\text{op}} \to \text{Sets}\) that can be represented as a filtered colimit of schemes along closed embeddings.

**Example 4.2.** \(A_\infty^{\text{ind}} := \bigcup_n A^n\) is an ind-scheme.

**Example 4.3.** The functor \(A^n((t)) : \text{AffSch}^{\text{op}} \to \text{Sets}\) sending \(\text{Spec} R\) to \(A^n((t))(\text{Spec} R) = A^n(R((t)))\) is represented by an ind-scheme.

**Proof.** We have

\[A^n((t))(\text{Spec} R) = A^n(R((t))) = R(t) \times^n = \text{colim}_k (t^{-k}R[[t]]) \times^n \text{colim} A^n[[t]](R)\]

Hence \(A^n((t))\) can be written as a colimit of \(A^n[[t]]\) along maps given by multiplication by powers of \(t\). \(\square\)

**Exercise 4.4.** For an affine scheme \(Y\) the functor \(Y((t)) : \text{AffSch}^{\text{op}} \to \text{Sets}\) sending \(\text{Spec} R\) to \(Y((t))(\text{Spec} R) = Y(R((t)))\) is represented by an ind-scheme.

**Definition 4.5.** We say that an ind-scheme is ind-affine if can be represented as a filtered colimit of affine schemes along closed embeddings.

**Remark 4.6.** Examples 4.2, 4.3 and 4.4 are ind-affine.

**Example 4.7.** The affine grassmannian \(\text{Gr}_G := G((t))/G[[t]]\) is an ind-scheme but it is not ind-affine.

**Definition 4.8.** For an ind-affine ind-scheme \(Z = \text{colim}_{i \in I} Z_i\) define \(C[Z] := \text{lim}_{i \in I} C[Z_i]\). This algebra carries a natural topology with \(\ker \alpha_i\), where \(\alpha_i : C[Z] \to C[Z_i]\), being neighborhoods of zero.

**Remark 4.9.** Given the algebra of functions \(A\) with topology one can recover the ind-affine ind-scheme as a colimit of \(\text{Spec}(A_j)\) for \(A \to A_j\) continuous where \(A_j\) has discrete topology.

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