OPERS I

EKATERINA BOGDANOVA

Recall that in [K][Theorem 3.3.1] we proved

Theorem 0.1. *There is a canonical isomorphism*

$$\mathfrak{z}(\hat{\mathfrak{sl}}_2)_x \cong \mathbb{C}[Proj(D_x)],$$

where $Proj(D_x)$ *is the space of projective connections on* $D_x := \mathrm{Spec}(\mathcal{O}_x)$.

In this note we will show that Theorem 0.1 implies

Theorem 0.2. *There is a canonical isomorphism*

$$\mathcal{Z}(\widetilde{U}_{\kappa_c}(\mathfrak{g})) =: \mathcal{Z}(\hat{\mathfrak{sl}}_2)_x \cong \mathbb{C}[Proj(\overset{\circ}{D}_x)],$$

where $Proj(\overset{\circ}{D}_x)$ *is the space of projective connections on* $\overset{\circ}{D}_x := \mathrm{Spec}(\mathcal{K}_x)$.

We will also generalize the statements in the following way. Let G be a simply-connected semi-simple algebraic group with the root system Δ. Let $\check{\Delta}$ be the dual root system.

Definition 0.1. Langlands dual group $\check{G}$ is the adjoint group with root system $\check{\Delta}$.

The goal of this seminar is to prove

Theorem 0.3. *There is a canonical isomorphism*

$$\mathfrak{z}(\hat{\mathfrak{g}})_x \cong \mathbb{C}[\mathrm{Op}_{\check{G}}(D_x)],$$

where $\mathrm{Op}_{\check{G}}(D_x)$ *is the space of opers, which will be analogs of projective connections for general* $\check{G}$.

Theorem 0.4. *There is a canonical isomorphism*

$$\mathcal{Z}(\hat{\mathfrak{g}})_x \cong \mathbb{C}[\mathrm{Op}_{\check{G}}(\overset{\circ}{D}_x)].$$

1. Generalities on connections.

Throughout this section let G be adjoint group. Let X be a smooth variety, $\mathcal{P}$ be a principal G-bundle. Abusing notation we will also denote the total space of this principal G-bundle by $\mathcal{P}$, and projection to X by f.

We have a G-equivariant short exact sequence

$$0 \to f^*\Omega^1_X \to \Omega^1_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}} \otimes \mathfrak{g}^* \to 0. \tag{1.1}$$

Definition 1.1. A connection ∇ on $\mathcal{P}$ is a G-equivariant section of (1.1).

Or, equivalently, a section $\mathcal{P} \times_G \mathfrak{g}^* \to (f_*\Omega^1_{\mathcal{P}})^G$ of

$$0 \to \Omega^1_X \to (f_*\Omega^1_{\mathcal{P}})^G \to \mathcal{P} \times_G \mathfrak{g}^* \to 0, \tag{1.2}$$

where $\mathcal{P} \times_G \mathfrak{g}^*$ is, by definition, the quotient of $\mathcal{P} \times \mathfrak{g}^*$ by the diagonal action.

Given a trivialization of $\mathcal{P}$ we get another section. Namely, then we get $\mathcal{P} \cong X \times G$ and therefore

$$\Omega^1_{\mathcal{P}} \cong f^*\Omega^1_X \oplus (\mathcal{O}_{\mathcal{P}} \otimes \mathfrak{g}^*).$$

Taking the difference we obtain an element of $\mathrm{Hom}(\mathcal{P} \times_G \mathfrak{g}^*, \Omega^1_X) \cong \mathrm{Hom}(\mathcal{O}_X, (\mathcal{P} \times_G \mathfrak{g}) \otimes \Omega^1_X)$.

Remark 1.2. Let X be a smooth 1-dimensional variety. Choose a local coordinate z. Then $\mathcal{P}$ trivializes and the data of the connection is an element of $\mathrm{Hom}(\mathcal{O}_X, \mathfrak{g} \otimes \Omega^1_X)$, and can be given by

$$\nabla = d_z + A(z)dz, \tag{1.3}$$

where A is a function valued in $\mathfrak{g}$. In this formula, d stands for the connection coming from the trivialization of $\mathcal{P}$.

Below we will abuse the notation and remove the dz from the formula (1.3). We will write a connection in the form

$$\partial_z + A(z).$$

Exercise 1.3. *Under change of trivialization of $\mathcal{P}$ by a function $g : X \to G$ we have*

$$\partial_z + A(z) \mapsto \partial_z + gA(z)g^{-1} - (\partial_z g)g^{-1}.$$

These are called gauge transformations*.*

Now let us specialize to the case $G = \mathrm{PGL}_2$. Choose a Borel $B \subset \mathrm{PGL}_2$. Suppose we have a connection ∇ on $\mathcal{P}$ and a section s of $\mathcal{P} \times_G (\mathrm{PGL}_2 / B)$. Note that PGL_2 / B is nothing but $\mathbb{P}^1$ with the natural action of PGL_2.

Remark 1.4. Giving a section $\mathcal{P} \times_G (\mathrm{PGL}_2 / B)$ is equivalent to giving a B-reduction $\mathcal{P}_B$ of $\mathcal{P}$. Indeed, for $\mathcal{P}_B$ take a fiber product

$$\begin{array}{ccc} \mathcal{P}_B & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{P} \times_G (\mathrm{PGL}_2 / B). \end{array}$$

Construction 1.5. Choose a local section s' of $\mathcal{P}$ lifting s. This gives a trivialization of $\mathcal{P}$ and hence an element of $\mathrm{Hom}(\mathcal{O}_X, \mathcal{P} \times_G \mathfrak{g} \otimes \Omega^1_X)$. Project to $\mathrm{Hom}(\mathcal{O}_X, \mathcal{P}_B \times_B \mathfrak{g}/\mathfrak{b} \otimes \Omega^1_X)$. The result is called *the derivative of s along* ∇. Note that it does not depend on s'.

2. Opers for PGL_2.

Set $X = \mathrm{Spec}(\mathbb{C}[[t]])$ or $\mathrm{Spec}(\mathbb{C}((t)))$.

Notation 2.1. Denote $D := \mathrm{Spec}(\mathbb{C}[[t]])$ and $\overset{\circ}{D} := \mathrm{Spec}(\mathbb{C}((t)))$.

Definition 2.2. A PGL_2-*oper* on X is a principal PGL_2-bundle $\mathcal{F}$ over X with a connection ∇, plus a globally defined section s of the associated $\mathbb{P}^1$-bundle $\mathcal{F} \times_{\mathrm{PGL}_2} (\mathrm{PGL}_2 / B)$, which has a nowhere vanishing derivative along ∇.

Remark 2.3. Note that although $\mathrm{Spec}(\mathbb{C}[[t]])$ and $\mathrm{Spec}(\mathbb{C}((t)))$ are not varieties, the constructions from previous sections still make sense.

For D set $\Omega^1_D := \mathbb{C}[[t]]dt$, and note that all G-bundles on D are trivial.

For $\overset{\circ}{D}$ set $\Omega^1_{\overset{\circ}{D}} := \mathbb{C}((t))dt$, and note by [S][III.2.3. Theorem $1'$] that all G-bundles on $\overset{\circ}{D}$ are trivial.

Remark 2.4. Definition 2.2 makes sense for a smooth curve as well, but we will specialize to X.

The condition that the derivative of s along ∇ is nowhere vanishing says that ∇ does not preserve $\mathcal{F}_B$ at any point. More precisely:

Exercise 2.5. *let $(\mathcal{F}, \nabla, s)$ be a PGL_2-oper. Choose a lift of s to $\mathcal{F}$ to get a trivialization of $\mathcal{F}$. Then*

$$\nabla_{\partial_t} = \partial_t + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},$$

where $a(t) + d(t) = 0$. The derivative of s along ∇ is nowhere vanishing if and only if $c(t)$ is nowhere vanishing (i.e. invertible element of $\mathbb{C}[[t]]$ or $\mathbb{C}((t))$).

Lemma 2.6. *Let $(\mathcal{F},\nabla,s)$ be as above. Then ∇ can be brought to a form*

$$\partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}$$

in a unique way. So we get that PGL_2*-opers non-canonically form an (ind)-affine space in the sense of algebraic geometry.*

Proof. Apply the gauge transformation by

$$\begin{pmatrix} c(t) & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2$$

to get a connection of the form

$$\partial_t + \begin{pmatrix} a_1(t) & b_1(t) \\ 1 & d_1(t) \end{pmatrix}.$$

Then apply the gauge transformation by

$$\begin{pmatrix} 1 & -a_1(t) \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2\,.$$

□

Remark 2.7. Another way to realize a PGL_2-oper $(\mathcal{F},\nabla,s)$ is the following. Trivialize the PGL_2-bundle $\mathcal{F}$ and choose a lift $\widetilde{\mathcal{F}}$ to an SL_2-torsor with connection ∇. The lift is unique up to tensoring with line bundles that square to $\mathcal{O}_X$ with connection.

Section of the associated $\mathbb{P}^1$-bundle s gives a B-reduction, which gives rize to a line subbundle $\mathcal{F}_1 \subset \widetilde{\mathcal{F}}$.

The connection ∇ on $\widetilde{\mathcal{F}}$ corresponds to a map $\mathcal{O}_X \to \widetilde{\mathcal{F}} \otimes \widetilde{\mathcal{F}}^* \otimes \Omega^1_X$, and thus to a map $\widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}} \otimes \Omega^1_X$. An oper condition means that for

$$0 \to \mathcal{F}_1 \xrightarrow{\nabla} \widetilde{\mathcal{F}} \otimes \Omega^1_X \to \widetilde{\mathcal{F}}/\mathcal{F}_1 \otimes \Omega^1_X \to 0$$

is an isomorphism.

This means that

$$0 \to \mathcal{F}_1 \otimes \Omega^1_X \otimes \widetilde{\mathcal{F}}/\mathcal{F}_1 \otimes \Omega^1_X \cong \mathcal{F}_1^2 \otimes \Omega^1_X \to 0,$$

and therefore

$$\mathcal{F}_1^2 \otimes \Omega^1_X \cong \det \widetilde{\mathcal{F}} \otimes (\Omega^1_X)^2,$$

i.e.

$$\mathcal{F}_1^2 \cong \Omega^1_X.$$

Choose $\mathcal{F}_1$ for a square root of Ω^1_X. We get

$$0 \to \Omega_X^{\frac{1}{2}} \to \widetilde{\mathcal{F}} \to \Omega_X^{-\frac{1}{2}} \to 0.$$

Proposition 2.8. *There is a one-to-one correspondence*

$$\{\mathrm{PGL}_2\text{ -}\textit{opers on } X\} \longleftrightarrow \{\textit{Projective connections on } X\}\,.$$

Proof. Let us canonically construct a projective connection from a PGL_2-oper. Let $(\widetilde{\mathcal{F}},\nabla)$ be as in Remark 2.7. We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^{\frac{1}{2}} & \xrightarrow[i]{} & \widetilde{\mathcal{F}} & \longrightarrow & \Omega_X^{-\frac{1}{2}} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\nabla} & & & & \\ 0 & \longrightarrow & \Omega_X^{\frac{3}{2}} & \longrightarrow & \widetilde{\mathcal{F}} \otimes \Omega^1_X & \xrightarrow[\pi]{} & \Omega_X^{\frac{1}{2}} & \longrightarrow & 0, \end{array}$$

such that $\pi \circ \nabla \circ i$ is the identity.

Let us construct a differential operator $\rho : \Omega_X^{-\frac{1}{2}} \to \Omega_X^{\frac{3}{2}}$. Let s be a section of $\Omega_X^{-\frac{1}{2}}$. Choose a lift s' of s to a section of $\mathcal{F}$. Then set

$$\rho(s) = \nabla(\tilde{s}),$$

where $\tilde{s} := s' - i \circ \pi \circ \nabla(s')$.

Now choose s such that $s^2 = (dt)^{-1}$ (the choice is unique up to a sign). Consider a basis $(s^{-1}, \tilde{s})$, and in this basis

$$\nabla_{\partial_t} = \partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}.$$

Exercise 2.9. *Using s and s^3 as trivializations of $\Omega_X^{\frac{1}{2}}$ and $\Omega_X^{\frac{3}{2}}$ respectively, show that $\rho = \partial_t^2 + v(t)$.*

So we constructed a canonical map which is seen to be a bijection. □

3. Opers for general G.

To define opers for general G we need to formulate an analog of the non-vanishing derivative condition.

Let G be an adjoint group. Choose $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. This gives $\mathfrak{n}_+ \subset \mathfrak{b} \subset \mathfrak{g}$ and

$$N = [B, B] \subset B \subset G \supset H.$$

Let f_i be the standard generators of $\mathfrak{n}_-$, let e_i be corresponding standard generators of $\mathfrak{n}_+$. Set $\mathfrak{n}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{n}_{-\alpha_i} = \mathbb{C}f_i$.

Let $[\mathfrak{n}, \mathfrak{n}]^\perp \subset \mathfrak{g}$ be the orthogonal complement of $[\mathfrak{n}, \mathfrak{n}]$ w.r.t. κ_0.

Lemma 3.1. *We have*

$$[\mathfrak{n}, \mathfrak{n}]^\perp / \mathfrak{b} \cong \oplus_{i=1}^{\operatorname{rk} \mathfrak{g}} \mathfrak{n}_{-\alpha_i}.$$

Construction 3.2. Note that B acts on $[\mathfrak{n}, \mathfrak{n}]^\perp / \mathfrak{b}$. There is a unique open B-orbit

$$\mathbb{O} \subset [\mathfrak{n}, \mathfrak{n}]^\perp / \mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$$

consisting of vectors with non-zero projection on each $\mathfrak{n}_{-\alpha_i}$. This orbit is isomorphic to B/N, where we use that G is adjoint.

Remark 3.3. Note that $\mathbb{O}$ only depends on the choice of $\mathfrak{b}$.

Construction 3.4. Recall that $X = \operatorname{Spec}(\mathbb{C}[[t]])$ or $\operatorname{Spec}(\mathbb{C}((t)))$. Let $\mathcal{F}$ be a G-torsor on X with a connection ∇ and B-reduction $\mathcal{F}_B$. Choose any flat connection ∇' on $\mathcal{F}$ preserving $\mathcal{F}_B$ (we can do it since $\mathcal{F}_B$ is trivial) and take $\nabla - \nabla'$. This gives a section of $\mathcal{F} \times_G \mathfrak{g} \otimes \Omega_X^1 \cong \mathcal{F}_B \times_B \mathfrak{g} \otimes \Omega_X^1$. Now project this section to $\mathcal{F}_B \times_B \mathfrak{g}/\mathfrak{b} \otimes \Omega_X^1$, and note that the result does not depend on ∇'. We call this element of $\operatorname{Hom}(\mathcal{O}_X, \mathcal{F}_B \times_B \mathfrak{g}/\mathfrak{b} \otimes \Omega_X^1)$ the *relative position of* ∇ *and* $\mathcal{F}_B$ and denote it by $\nabla/\mathcal{F}_B$.

Definition 3.5. We say that $\mathcal{F}_B$ is *transversal* to ∇ if

$$\nabla/\mathcal{F}_B \in \operatorname{Hom}(\mathcal{O}_X, \mathcal{F}_B \times_B \mathbb{O} \otimes \Omega_X^1) \subset \operatorname{Hom}(\mathcal{O}_X, \mathcal{F}_B \times_B \mathfrak{g}/\mathfrak{b} \otimes \Omega_X^1).$$

Remark 3.6. When $G = \mathrm{PGL}_2$ this is exactly the non-vanishing condition we had before.

Definition 3.7. A *G-oper* on X is a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal G-bundle, ∇ is a connection on $\mathcal{F}$ such that $\mathcal{F}_B$ is transversal to ∇.

Remark 3.8. Choose a trivialization of $\mathcal{F}_B$. Then the oper condition says that

$$\nabla_{\partial_t} = \partial_t + \sum_{i=1}^{\operatorname{rk} \mathfrak{g}} \psi_i(t) f_i + v(t), \tag{3.1}$$

where ψ_i are non-vanishing functions (i.e. invertible elements of $\mathbb{C}[[t]]$ or $\mathbb{C}((t))$) , and $v(t)$ is a $\mathfrak{b}$-valued function.

Definition 3.9. Let $\widetilde{\mathrm{Op}}_G(X)$ be the space of opers of the form

$$\nabla_{\partial_t} = \partial_t + \sum_{i=1}^{\operatorname{rk} \mathfrak{g}} f_i + v(t), \tag{3.2}$$

where $v(t)$ is a $\mathfrak{b}$-valued function.

Notation 3.10. For an affine scheme T we denote by T_X the arc scheme $T[[t]]$ in the case of $X = D$ and the loop ind-scheme $T((t))$ in the case of $X = \overset{\circ}{D}$ (see section 4).

There is a natural action of N_X on $\widetilde{\mathrm{Op}}_G(X)$.

Lemma 3.11. *We have*

$$\mathrm{Op}_G(X) \cong \widetilde{\mathrm{Op}}_G(X)/N_X.$$

Proof. Recall that B-orbit $\mathbb{O}$ is an H-torsor, so we can bring a connection of the form (3.1) to the form (3.2) by a unique element of H_X. □

We are now going to describe the (ind)-scheme of opers using the Kostant slice.

Notation 3.12. Let $p_{-1} = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} f_i$, and let $(p_{-1}, 2\check{\rho}, p_1)$ be the principal $\mathfrak{sl}_2$-triple.

Note that $\ker \mathrm{ad}\, p_{-1} \subset \mathfrak{b}$, and $p_{-1} + \mathfrak{b}$ is stable under the action of N. Recall that the following two proposition and their corollaries were discussed in [K]:

Proposition 3.13. *(Kostant) The map*

$$N \times S \to p_{-1} + \mathfrak{b}$$

given by the action is an isomorphism.

Corollary 3.14. $N_X \times S_X \cong (p_{-1} + \mathfrak{b})_X$.

Proposition 3.15. *(Kostant) The composition of the embedding $S \hookrightarrow \mathfrak{g}$ and the quotient morphism $\mathfrak{g} \to \mathfrak{g}//G$ is an isomorphism.*

Corollary 3.16. $S_X \to \mathfrak{g}_X//G_X$.

We now apply them in the study of opers.

Proposition 3.17. *The morphism of schemes*

$$N[[t]] \times \{\partial_t + S[[t]]\} \to \{\partial_t + (p_{-1} + \mathfrak{b})[[t]]\} = \widetilde{\mathrm{Op}}_G(D) \tag{3.3}$$

induced by the gauge transformation action is an isomorphism.

Proof. Let $\mathbb{A}^1_\lambda = \mathrm{Spec}(\mathbb{C}[\lambda])$ and let $\mathrm{AffSch}_{/\mathbb{A}^1_\lambda}$ be the category of affine schemes over $\mathbb{A}^1_\lambda$. Consider $\{\lambda\partial_t + S[[t]]\}$, $\{\lambda\partial_t + (p_{-1} + \mathfrak{b})[[t]]\} \in \mathrm{AffSch}_{/\mathbb{A}^1_\lambda}$.

By definition, the gauge action of $n(t) \in N[[t]]$ on $(\lambda\partial_t + s(t)) \in \{\lambda\partial_t + S[[t]]\}$ is given by

$$n(t) \cdot (\lambda\partial_t + s(t)) = \lambda\partial_t + \mathrm{Ad}(n(t))s(t) - \lambda(\partial_t n(t))n(t)^{-1}. \tag{3.4}$$

Note that $N[[t]] \times \{\lambda\partial_t + S[[t]]\}$ and $\{\lambda\partial_t + (p_{-1} + \mathfrak{b})[[t]]\}$ admit a $\mathbb{G}_m$-action such that the action map is equivariant. Indeed, for $z \in \mathbb{C}^\times$ we have

$$z \cdot (n(t), \lambda\partial_t + s(t)) := (z\,\mathrm{Ad}(\check{\rho}(z))n(t), z\lambda\partial_t + z\,\mathrm{Ad}\,\check{\rho}(z)s(t)),$$

$$z\cdot(\lambda\partial_t + \mathrm{Ad}(n(t))s(t) - \lambda(\partial_t n(t))n(t)^{-1}) = z\lambda\partial_t + z\,\mathrm{Ad}(\check{\rho}(z))\,\mathrm{Ad}(n(t))s(t) + z\lambda\,\mathrm{Ad}(\check{\rho}(z))(\partial_t n(t))n(t)^{-1}.$$

The restriction of the action map to the fiber $\lambda = 1$ is the desired map (3.3). This implies that there is a natural filtration on $\mathbb{C}[N[[t]]] \otimes \mathbb{C}[\{\partial_t + S[[t]]\}]$ and $\mathbb{C}[\{\partial_t + (p_{-1} + \mathfrak{b})[[t]]\}]$ such that the pullback map

$$\mathbb{C}[\{\partial_t + (p_{-1} + \mathfrak{b})[[t]]\}] \to \mathbb{C}[N[[t]]] \otimes \mathbb{C}[\{\partial_t + S[[t]]\}] \tag{3.5}$$

is filtered. Note that the filtration is non-negative. However, the restriction of the action map to the fiber $\lambda = 0$ is the isomorphism from Corollary 3.14. Thus the associated graded of (3.5) is an isomorphism, and hence (3.5) is also an isomorphism. □

Corollary 3.18. *We have*

$$\mathrm{Op}_G(D) \cong \{\partial_t + S[[t]]\}.$$

Moreover, we get that

$$\mathrm{gr}\,\mathbb{C}[\mathrm{Op}_G(D)] \cong \mathbb{C}[S[[t]]] \cong \mathbb{C}[\mathfrak{g}[[t]]]^{G[[t]]},$$

and hence $\mathbb{C}[\mathrm{Op}_G(D)] \cong \mathbb{C}[\mathfrak{g}[[t]]]^{G[[t]]}$.

We also have an analogous statement for loop spaces defined in Example 4.3:

Proposition 3.19. *The morphism of ind-schemes*

$$N((t)) \times \{\partial_t + S((t))\} \to \{\partial_t + (p_{-1} + \mathfrak{b})((t))\} = \widetilde{\mathrm{Op}}_G(\overset{\circ}{D}) \tag{3.6}$$

induced by the gauge transformation action is an isomorphism.

Corollary 3.20. *We have an isomorphism of ind-schemes*

$$\mathrm{Op}_G(\overset{\circ}{D}) \cong \{\partial_t + S((t))\}.$$

Finally, to make sense of the statement of Theorem 0.4 we need define algebras of functions on ind-schemes ($\mathrm{Op}_G(\overset{\circ}{D})$ in our case). This is addressed in Definition 4.8.

4. Appendix: Ind-schemes and loop spaces.

Definition 4.1. An ind-scheme is a functor $\mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ that can be represented as a filtered colimit of schemes along closed embeddings.

Example 4.2. $\mathbb{A}^{\infty}_{\mathrm{ind}} := \cup_n \mathbb{A}^n$ *is an ind-scheme.*

Example 4.3. *The functor* $\mathbb{A}^n((t)) : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ *sending* $\operatorname{Spec} R$ *to* $\mathbb{A}^n((t))(\operatorname{Spec} R) = \mathbb{A}^n(R((t)))$ *is represented by an ind-scheme.*

Proof. We have

$$\mathbb{A}^n((t))(\operatorname{Spec} R) = \mathbb{A}^n(R((t))) = R((t))^{\times n} = \mathrm{colim}_k (t^{-k} R[[t]])^{\times n} = \mathrm{colim} \mathbb{A}^n[[t]](R).$$

Hence $\mathbb{A}^n((t))$ can be written as a colimit of $\mathbb{A}^n[[t]]$ along maps given by multiplication by powers of t. □

Exercise 4.4. *For an affine scheme* Y *the functor* $Y((t)) : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ *sending* $\operatorname{Spec} R$ *to* $Y((t))(\operatorname{Spec} R) = Y(R((t)))$ *is represented by an ind-scheme.*

Definition 4.5. We say that an ind-scheme is *ind-affine* if can be represented as a filtered colimit of affine schemes along closed embeddings.

Remark 4.6. Examples 4.2, 4.3 and 4.4 are ind-affine.

Example 4.7. *The affine grassmannian* $\mathrm{Gr}_G := G((t))/G[[t]]$ *is an ind-scheme but it is not ind-affine.*

Definition 4.8. For an ind-affine ind-scheme $Z = \mathrm{colim}_{i \in I} Z_i$ define $\mathbb{C}[Z] := \lim_{i \in I^{op}} \mathbb{C}[Z_i]$. This algebra carries a natural topology with $\ker \alpha_i$, where $\alpha_i : \mathbb{C}[Z] \to \mathbb{C}[Z_i]$, being neighborhoods of zero.

Remark 4.9. Given the algebra of functions A with topology one can recover the ind-affine ind-scheme as a colimit of $\mathrm{Spec}(A_j)$ for $A \to A_j$ continuous where A_j has discrete topology.

References

[B] "Opers II." E. Bogdanova, seminar notes
[K] "Invariants of jets and the center for $\hat{\mathfrak{sl}}_2$." I. Karpov, seminar notes
[S] "Galois cohomology." J.-P. Serre
[W] "Central elements of the completed universal enveloping algebra." H. Wan, seminar notes

Harvard University, USA
Email address: ebogdanova@math.harvard.edu