OPERS II

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Set $X = \mathrm{Spec}(\mathbb{C}[[t]])$ or $\mathrm{Spec}(\mathbb{C}((t)))$.

Notation 0.1. *We will denote* $D := \mathrm{Spec}(\mathbb{C}[[t]])$ *and* $\overset{\circ}{D} := \mathrm{Spec}(\mathbb{C}((t)))$.

Notation 0.2. *For an affine scheme* Y *by* Y_X *we mean either the jet scheme* JY *or the loop ind-scheme* LY.

Recall that if $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a G-oper on X then

$$\nabla_{\partial_t} = \partial_t + \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \psi_i(t) f_i + v(t),$$

where $\psi_i(t) \neq 0$ for all i and $v(t) \in \mathfrak{b}_X$.

Therefore

$$\mathrm{Op}_G(X) \cong \{\partial_t + \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \psi_i(t) f_i + v(t),\ \psi_i(t) \neq 0,\ v(t) \in B_X\}/B_X.$$

Since every oper of the form

$$\nabla_{\partial_t} = \{\partial_t + \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \psi_i(t) f_i + v(t),\ \psi_i(t) \neq 0,\ v(t) \in B_X\}$$

can be represented in the form

$$\{\partial_t + \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} f_i + v(t),\ v(t) \in B_X\}$$

by gauging by a unique element of H_X, we get that

$$\mathrm{Op}_G(X) \cong \widetilde{\mathrm{Op}}_G(X)/N_X. \tag{0.1}$$

In the first part of this talk we also proved that

$$\mathrm{Op}_G(X) \cong \widetilde{\mathrm{Op}}_G(X)/N_X \cong \{\partial_t + S_X\}, \tag{0.2}$$

where S is the Kostant slice.

1. Action of coordinate changes.

In this section we want to see how $\mathrm{Aut}(\mathcal{O})$ acts on the canonical representatives from (0.2). Let s be another coordinate on D, i.e. $s = \sum_{i \geq 1} a_i t^i$ with $a_1 \neq 0$. Let $t = \phi(s)$.

Recall that $p_{-1} := \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} f_i$, and $(p_{-1}, 2\check{\rho}, p_1)$ is the principal $\mathfrak{sl}_2$-triple. Set $V := \ker \mathrm{ad}\, p_1$ (note that Kostant slice $S = p_{-1} + V$). The operator $\mathrm{ad}\,\check{\rho}$ defines a grading on $\mathfrak{g} = \oplus_i \mathfrak{g}_i$.

Let $d_i, \ldots, d_{\mathrm{rk}\,\mathfrak{g}}$ denote the exponents ($d_i + 1$ are the degrees of free homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$) of $\mathfrak{g}$. Then

$$V = \oplus_{i=1}^{\mathrm{rk}\,\mathfrak{g}} V_{d_i}.$$

Note that p_1 spans $V_{d_1} = V_1$. Choose p_j to be linear generator of V_{d_j}.

Using (0.2) write a G-oper as

$$\nabla_{\partial_t} = \partial_t + p_{-1} + v(t),\ v(t) \in V_X.$$

Write $v(t) = \sum_{j=1}^{\mathrm{rk}\,\mathfrak{g}} v_j(t) p_j$ for $v_j(t) \in \mathbb{C}[X]$.

Then

$$\nabla_{\partial_t} = \nabla_{\phi'(s)^{-1}\partial_s} = \phi'(s)^{-1}\partial_s + p_{-1} + v(\phi(s)),$$

hence

$$\nabla_{\partial_s} = \partial_s + \phi'(s)p_{-1} + \phi'(s)v(\phi(s)). \tag{1.1}$$

We now need to apply gauge transformations to bring this connection to the canonical form from (0.2).

We first apply gauge transformation by $\check{\rho}(\phi'(s))$, where

$$\check{\rho} = \sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \check{\omega}_i : \mathbb{C}^\times \to H.$$

Under this gauge transformation (1.1) becomes

$$\partial_s + p_{-1} + \tilde{v}(s),$$

where

$$\tilde{v}(s) := \phi'(s)\check{\rho}(\phi'(s))v(\phi(s))\check{\rho}(\phi'(s))^{-1} - d\check{\rho}(\frac{\phi''(s)}{\phi'(s)}).$$

Note that this is an element of $\widetilde{\mathrm{Op}}_G(X)$, and by (0.1) there exist unique $g \in N_X$ and $\partial_s + p_{-1} + \bar{v}(s)$ with $\bar{v}(s) \in V_X$ such that

$$\partial_s + p_{-1} + \bar{v}(s) = g \cdot (\partial_s + p_{-1} + \tilde{v}(s)),$$

Exercise 1.1. *Find that*

(1) $g = \exp(\frac{1}{2}\frac{\phi''}{\phi'}p_1)$ *Hint: for* $s \in S$, *find* $g \in G$ *such that* $\mathrm{Ad}\, s(\check{\rho} + s) \in S$.,
(2) $\bar{v}_1 = v_1(\phi(s))(\phi')^2 - \frac{1}{2}\{\phi, s\}$, *where* $\{\phi, s\} := \frac{\phi'''}{\phi'} - \frac{3}{2}(\frac{\phi''}{\phi'})^2$,
(3) $\bar{v}_j = v_j(\phi(s))(\phi')^{d_j+1}$ *for* $j > 1$.

So we defined the action of $\mathrm{Aut}(\mathcal{O})$ on $\mathrm{Op}_G(X)$ and therefore can form $\mathrm{Op}_G(D_x)$ and $\mathrm{Op}_G(\overset{\circ}{D}_x)$.

The formulae from Exercise 1.1 show that under changes of coordinates v_1 transforms as a projective connection and v_j for $j > 1$ transform as $(d_j + 1)$-differential forms on D_x or $\overset{\circ}{D}_x$. Hence

$$\mathrm{Op}_G(D_x) \cong \mathit{Proj}(D_x) \times \oplus_{j=2}^{\mathrm{rk}\,\mathfrak{g}} \Omega_{\mathcal{O}_x}^{\otimes(d_j+1)},$$

$$\mathrm{Op}_G(\overset{\circ}{D}_x) \cong \mathit{Proj}(\overset{\circ}{D}_x) \times \oplus_{j=2}^{\mathrm{rk}\,\mathfrak{g}} \Omega_{\mathcal{K}_x}^{\otimes(d_j+1)}.$$

2. The center for the arbitrary Kac-Moody algebra.

Recall that the main goal of the seminar is to prove the following two statements:

Theorem 2.1. *There is a canonical isomorphism*

$$\mathfrak{z}(\hat{\mathfrak{g}})_x \cong \mathbb{C}[\mathrm{Op}_G(D_x)].$$

Or, equivalently, there is a canonical $(\mathrm{Der}(\mathcal{O}), \mathrm{Aut}(\mathcal{O}))$*-equivariant isomorphism*

$$\mathfrak{z}(\hat{\mathfrak{g}}) \cong \mathbb{C}[\mathrm{Op}_G(D)].$$

Theorem 2.2. *There is a canonical isomorphism*

$$\mathcal{Z}(\hat{\mathfrak{g}})_x \cong \mathbb{C}[\mathrm{Op}_G(\overset{\circ}{D}_x)].$$

Or, equivalently, there is a canonical $(\mathrm{Der}(\mathcal{O}), \mathrm{Aut}(\mathcal{O}))$*-equivariant isomorphism*

$$\mathcal{Z}(\hat{\mathfrak{g}}) \cong \mathbb{C}[\mathrm{Op}_G(\overset{\circ}{D})].$$

The goal of this section is to deduce Theorem 2.2 from Theorem 2.1.

Remark 2.1. The second isomorphism in Theorem 2.1 is compatible with the natural filtrations on both sides. Recall that the filtration on $\mathfrak{z}(\hat{\mathfrak{g}})$ is induced from the PBW filtration on $V_{\kappa_c}(\mathfrak{g})$. And the filtration on $\mathbb{C}[\mathrm{Op}_G(D_x)]$ was introduced in the proof of (0.2) in [B][Proposition 3.17].

This implies the following statement:

Lemma 2.2. *The natural embedding* $\operatorname{gr}\mathfrak{z}(\hat{\mathfrak{g}}) \hookrightarrow \mathbb{C}[J\mathfrak{g}]^{JG}$ *is an isomorphism.*

Let $f_1, \dots f_{\mathrm{rk}\,\mathfrak{g}}$ be free homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$. Let $f_{i,n}$, $1 \le i \le \mathrm{rk}\,\mathfrak{g}$, $n < 0$ be corresponding free homogeneous generators of $\mathbb{C}[J\mathfrak{g}]^{JG}$ (see [W][Section 3.4] and [K][Theorem 1.3.1]).

Remark 2.3. Under the second isomorphism in Theorem 2.1 the element $f_{1,-1} \in \mathbb{C}[\mathrm{Op}_G(D)]$ goes to the Segal-Sugawara vector $S_{-2} = \frac{1}{2}\sum_i x_i(-1)x^i(-1)|0\rangle$. Using equivariance of the isomorphism under the action of $L_{-1} = -\partial_t \in \mathrm{Der}(\mathcal{O})$ we get that $f_{1,-k} \in \mathbb{C}[\mathrm{Op}_G(D)]$ goes to the Segal-Sugawara vector S_{-k-1}.

Recall from [W][Remark 3.9] that we have a homomorphism

$$\widetilde{U}(\mathfrak{z}(\hat{\mathfrak{g}})) \to \mathcal{Z}(\hat{\mathfrak{g}}).$$

Proposition 2.4. *The homomorphism* $\widetilde{U}(\mathfrak{z}(\hat{\mathfrak{g}})) \to \mathcal{Z}(\hat{\mathfrak{g}})$ *is an isomorphism.*

Proof. Recall that $\mathfrak{z}(\hat{\mathfrak{g}}) \cong \mathbb{C}[f_{i,n}]$ for $1 \le i \le \mathrm{rk}\,\mathfrak{g}$, $n < 0$. By [L] we have

$$\widetilde{U}(\mathfrak{z}(\hat{\mathfrak{g}})) \cong \mathbb{C}[L(\mathfrak{g} /\!\!/ G)] = \lim_{N \in \mathbb{Z}_+} \mathbb{C}[f_{i,n}, 1 \le i \le \mathrm{rk}\,\mathfrak{g}, n \in \mathbb{Z}]/(f_{i,n}, n > N).$$

The latter is equivalent to

$$\lim_{N \in \mathbb{Z}_+} \mathbb{C}[f_{i,n}, 1 \le i \le \mathrm{rk}\,\mathfrak{g}, n \in \mathbb{Z}]/(f_{i,n}, n > (d_i+1)N) \cong \lim_{N \in \mathbb{Z}_+} (\mathbb{C}\left[\mathfrak{g} \otimes t^{-N}\mathbb{C}[[t]]\right]^{JG}).$$

It suffices to prove that

$$\mathbb{C}\left[\mathfrak{g} \otimes t^{-N}\mathbb{C}[[t]]\right]^{JG} \to \mathcal{Z}(\hat{\mathfrak{g}})/\mathcal{Z}(\hat{\mathfrak{g}}) \cap I_N, \tag{2.1}$$

where $I_N := (\mathfrak{g} \otimes t^N\mathbb{C}[[t]])$, is an isomorphism. PBW filtration on $U_{\kappa_c}(\hat{\mathfrak{g}})$ induces a filtration on $\mathcal{Z}(\hat{\mathfrak{g}})/\mathcal{Z}(\hat{\mathfrak{g}}) \bigcap I_N$ (note that we need to take the quotient since the filtration on $\mathcal{Z}(\hat{\mathfrak{g}})$ is not exhaustive), and (2.1) is filtered. Hence it suffices to check that

$$\operatorname{gr}(\mathbb{C}\left[\mathfrak{g} \otimes t^{-N}\mathbb{C}[[t]]\right]^{JG}) \to \operatorname{gr}(\mathcal{Z}(\hat{\mathfrak{g}})/\mathcal{Z}(\hat{\mathfrak{g}}) \cap I_N) \tag{2.2}$$

is an isomorphism.

But since $U_{\kappa_c}(\hat{\mathfrak{g}})/I_N$ is $J\mathfrak{g}$-stable we have

$$\operatorname{gr}(\mathcal{Z}(\hat{\mathfrak{g}})/\mathcal{Z}(\hat{\mathfrak{g}}) \cap I_N) \hookrightarrow \operatorname{gr}\left((U_{\kappa_c}(\hat{\mathfrak{g}})/I_N)^{J\mathfrak{g}}\right) \hookrightarrow (\operatorname{gr} U_{\kappa_c}(\hat{\mathfrak{g}})/\operatorname{gr} I_N)^{J\mathfrak{g}}. \tag{2.3}$$

Note that

$$\operatorname{gr} U_{\kappa_c}(\hat{\mathfrak{g}})/\operatorname{gr} I_N \cong \mathrm{Sym}\,\mathfrak{g}((t))/(\mathfrak{g} \otimes t^N\mathbb{C}[[t]]) \cong \mathbb{C}\left[\mathfrak{g}^* \otimes t^{-N}\mathbb{C}[[t]]\right] \cong \mathbb{C}\left[\mathfrak{g} \otimes t^{-N}\mathbb{C}[[t]]\right],$$

and therefore

$$(\operatorname{gr} U_{\kappa_c}(\hat{\mathfrak{g}})/\operatorname{gr} I_N)^{J\mathfrak{g}} \cong \mathbb{C}\left[\mathfrak{g} \otimes t^{-N}\mathbb{C}[[t]]\right]^{JG}. \tag{2.4}$$

So we get that the composition of (2.2) and (2.4)∘(2.3) is identity. Since (2.4)∘(2.3) is injective left inverse to (2.2) we get that (2.2) is an isomorphism.

□

Corollary 2.5. $\mathcal{Z}(\hat{\mathfrak{g}}) \cong \widetilde{U}\left(\mathbb{C}[\mathrm{Op}_G(D)]\right)$.

Exercise 2.6. $\widetilde{U}(\mathbb{C}[\mathrm{Op}_G(D)]) \cong \mathbb{C}[\mathrm{Op}_G(\overset{\circ}{D})]$.

So we deduced Theorem 2.2.

Proposition 2.7. *For* $\kappa \neq \kappa_c$ *one has* $\mathcal{Z}(\widetilde{U}_\kappa(\hat{\mathfrak{g}})) \cong \mathbb{C}$.

References

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