PARABOLIC WAKIMOTO MODULES AND APPLICATIONS

KENTA SUZUKI

We will define generalized Wakimoto modules, which gives a functorial way of constructing $\widehat{\mathfrak{g}}$-modules from $\widehat{\mathfrak{m}}$-modules for parabolic subalgebras $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{u} \subset \mathfrak{g}$. We will give applications of Wakimoto modules, including the Kac-Kazhdan conjecture, which computes the characters of Verma modules $\mathbb{M}_{\lambda,\kappa_c}$ on the critical level for $\lambda \in \mathfrak{h}^*$ *generic*, i.e., not lying in a countable union of hyperplanes.

1. Semi-infinite parabolic induction

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra with Borel subalgebra $\mathfrak{b}_+$ and Cartan subalgebra $\mathfrak{h}$ (we work in this generality since we want to apply our construction to Levi subalgebras of simple Lie algebras).

Wakimoto modules, constructed in [Wan24b], are the images of Fock modules under a functor $\widetilde{U}_\kappa(\mathfrak{h})-\text{mod} \to \widetilde{U}_{\kappa+\kappa_c}(\mathfrak{g})-\text{mod}$.[1] We want to generalize the construction by replacing the Borel subalgebra $\mathfrak{b}$ with an arbitrary parabolic subalgebra $\mathfrak{p}$ and replacing the Cartan subalgebra $\mathfrak{h}$ with the Levi component $\mathfrak{m}$ of $\mathfrak{p}$. Let us first recall what a parabolic subalgebra is:

Definition 1.1. A *parabolic subalegbra* is a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that one of the following equivalent conditions hold:

- $\mathfrak{p}$ contains a Borel subalgebra of $\mathfrak{g}$; or
- the orthogonal complement of $\mathfrak{p}$ with respect to an invariant orthogonal form[2] is its nilradical.

Example 1.2. $\mathfrak{b}_+$ and $\mathfrak{g}$ are parabolic subalgebras of $\mathfrak{g}$.

Each conjugacy class of parabolic subalgebras has a unique representative containing $\mathfrak{b}_+$: we call those parabolic subalgebras *standard*. Let Δ_s be the set of simple roots corresponding to $\mathfrak{b}_+ \subset \mathfrak{g}$. Then standard parabolic subalgebras of $\mathfrak{g}$ are classified by subsets of Δ_s: so $\mathfrak{b}_+$ corresponds to $\varnothing$ and $\mathfrak{g}$ corresponds to Δ_s. More generally, for a subset $S \subset \Delta_s$, the corresponding *standard parabolic subalgebra* $\mathfrak{p}_S \subset \mathfrak{g}$ is

$$\mathfrak{p}_S := \mathfrak{b}_+ \oplus \bigoplus_{\substack{\alpha>0 \\ \alpha \in \operatorname{span} \Delta_s}} \mathfrak{g}_\alpha.$$

The Levi component is then given by:

$$\mathfrak{m}_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in \operatorname{span} \Delta_s} \mathfrak{g}_\alpha.$$

Analogous to the opposite Borel subalgebra, let

$$\mathfrak{p}_{S,-} := \mathfrak{b}_- \oplus \bigoplus_{\substack{\alpha<0 \\ \alpha \in \operatorname{span} \Delta_s}} \mathfrak{g}_\alpha$$

be the *opposite parabolic*.

[1] These are categories of smooth modules.

[2] When $\mathfrak{g}$ is semisimple we can use the Killing form, but for arbitrary reductive Lie algebras the Killing form may be degenerate.

Example 1.3. When $\mathfrak{g} = \mathfrak{sl}_n$, let S be a subset of $\Delta_s = \{\alpha_1, \dots, \alpha_{n-1}\}$ such that $\Delta_s \backslash S = \{\alpha_{a_1}, \dots, \alpha_{a_k}\}$. The corresponding parabolic subalgebras are

$$\mathfrak{p}_S = \mathfrak{sl}_n \cap \begin{pmatrix} M_{a_1 \times a_1} & * & * & * \\ 0 & M_{(a_2-a_1)\times(a_2-a_1)} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & M_{(n-a_k)\times(n-a_k)} \end{pmatrix}$$

and

$$\mathfrak{p}_{S,-} = \mathfrak{sl}_n \cap \begin{pmatrix} M_{a_1 \times a_1} & & & \\ * & M_{(a_2-a_1)\times(a_2-a_1)} & & \\ * & * & \ddots & \\ * & * & * & M_{(n-a_k)\times(n-a_k)} \end{pmatrix}.$$

The Levi component is

$$\begin{aligned} \mathfrak{m}_S &= \{(x_0, \dots, x_k) \in \mathfrak{gl}_{a_1} \times \cdots \times \mathfrak{gl}_{n-a_k} : \mathrm{tr}(x_0) + \cdots + \mathrm{tr}(x_k) = 0\} \\ &\simeq \mathfrak{sl}_{a_1} \times \cdots \times \mathfrak{sl}_{n-a_k} \times \mathbb{C}^{\oplus k}. \end{aligned}$$

First, we must extend relevant definitions from simple Lie algebras to reductive Lie algebras. The following is the generalization of the critical level:

Definition 1.4. Let $\mathfrak{g}$ be a reductive Lie algebra, which decomposes as $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i \oplus \mathfrak{g}_0$ for some simple Lie algebras $\mathfrak{g}_1, \dots, \mathfrak{g}_s$ and an abelian Lie algebra $\mathfrak{g}_0$. Then the *critical level* is $\kappa_c(\mathfrak{g}) := (\kappa_{i,c})_{i=0}^s$, where $\kappa_{0,c} = 0$ and $\kappa_{i,c}$ is the critical level for the simple Lie algebra $\mathfrak{g}_i$ for $1 \le i \le s$.

Given an invariant symmetric bilinear form κ on $\mathfrak{g}$, let $\widehat{\mathfrak{g}}_\kappa$ be the corresponding affine Kac-Moody algebra, as in [KL24]: it is the central extension

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}(\!(t)\!) \to 0$$

with commutation relation

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - \big(\kappa(A, B)\mathrm{Res} f dg\big)\mathbf{1}.$$

Let us now formally re-state our goal:

Goal 1.5. Let $\mathfrak{g}$ be a reductive Lie algebra, let κ be an invariant symmetric bilinear form on $\mathfrak{g}$, and let $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{u} \subset \mathfrak{g}$ be a parabolic subalgebra. Define an exact functor

$$\widetilde{U}_{\kappa|_{\mathfrak{m}} + \kappa_c(\widehat{\mathfrak{m}})}(\mathfrak{m})\text{−mod} \to \widetilde{U}_{\kappa + \kappa_c}(\widehat{\mathfrak{g}})\text{−mod}$$

such that the Wakimoto module with highest weight λ is sent to the Wakimoto module with highest weight λ.

1.1. **Finite-dimensional analog.** Let us first describe the finite-dimensional analog of Goal 1.5.

Definition 1.6. Let $\mathfrak{g}$ be a simple Lie algebra with standard parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \ltimes \mathfrak{u}$. There is an exact functor, the *parabolic induction functor*

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \colon \mathfrak{m}\text{−mod} \to \mathfrak{g}\text{−mod}.$$

Given a $\mathfrak{m}$-module V, we may view it as a $\mathfrak{p}$-module by extension by zero, i.e., by making $\mathfrak{u}$ act by zero, and we let

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V.$$

Now the $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ sends Verma modules to Verma modules:

Lemma 1.7. *For a weight $\lambda \in \mathfrak{h}^*$, let $V_{\mathfrak{m}}(\lambda)$ and $V_{\mathfrak{g}}(\lambda)$ be the Verma modules with highest weight λ of the Lie algebras $\mathfrak{m}$ and $\mathfrak{g}$, respectively. Then*

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V_{\mathfrak{m}}(\lambda) \simeq V_{\mathfrak{g}}(\lambda).$$

Proof. Follows from observing that $U(\mathfrak{p}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda$ is isomorphic to the inflation of the $\mathfrak{m}$-module $V_\mathfrak{m}(\lambda)$ to $\mathfrak{p}$, and because induction is transitive. □

Remark 1.8. When $\mathfrak{p} = \mathfrak{b}_+$, the above recovers the construction of Verma modules (i.e., $V_\mathfrak{g}(\lambda) = \mathrm{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}_\lambda$).

Recall that [Kiy24] gives a geometric construction of dual Verma modules using vector fields on G/N_-, where N_- is the unipotent radical of the opposite Borel subalgebra B_-. The construction admits a straightforward generalization to the parabolic setting: let $P_\pm = M \ltimes U_\pm \subset G$ be subgroups whose Lie algebras are $\mathfrak{p}_\pm = \mathfrak{m} \ltimes \mathfrak{u}_\pm \subset \mathfrak{g}$. Then analogously to [Kiy24, §2] there is a map of Lie algebras

$$\mathfrak{g} \to \mathrm{Vect}(G/U_-)^{M_r},$$

where M_r acts on G/U_- from the right.[3] Now as in Daishi's talk, $P_+U_-/U_- \subset G/U_-$ is Zariski open, and restricting to the locus gives a homomorphism of algebras

$$\varphi_{P_+}^G : U(\mathfrak{g}) \to D(P_+)^M \simeq D(U_+) \otimes U(\mathfrak{m}), \tag{1.9}$$

where the second isomorphism follows from the isomorphism of varieties $P_+ \simeq U_+ \times M$. Now:

Lemma 1.10. *Let V be a $\mathfrak{m}$-module, with structure morphism $\varphi\colon U(\mathfrak{m}) \to \mathrm{End}(V)$. Then the modified $\mathfrak{g}$-module structure on $\mathbb{C}[U_+] \otimes V$ is defined by*

$$U(\mathfrak{g}) \to D(U_+) \otimes U(\mathfrak{m}) \xrightarrow{1\otimes\varphi} D(U_+) \otimes \mathrm{End}(V) \to \mathrm{End}(\mathbb{C}[U_+] \otimes V),$$

noting that $\mathbb{C}[U_+]$ is naturally a $D(U_+)$-module. Then the $\mathfrak{g}$-module $\mathbb{C}[U_+] \otimes V^\vee$ is isomorphic to the dual parabolic induction $(\mathrm{Ind}_\mathfrak{p}^\mathfrak{g} V)^\vee$.[4]

We hope to see Lemma 1.7 from the geometric perspective:

Proposition 1.11. *Let $P_+ = M \ltimes U_+ \subset G$ be a standard parabolic subgroup. There is a commutative diagram*

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[r, "\varphi_{B_+}^G"] \arrow[d, "\varphi_{P_+}^G"] & D(N_+) \otimes U(\mathfrak{h}) \arrow[d, "\simeq"] \\ D(U_+) \otimes U(\mathfrak{m}) \arrow[r, "\mathrm{id}_{D(U_+)} \otimes \varphi_{B_+\cap M}^M"] & D(U_+) \otimes \big(D(N_+ \cap M) \otimes U(\mathfrak{h})\big). \end{tikzcd}$$

Here, the homomorphisms $U(\mathfrak{g}) \to D(N_+) \otimes U(\mathfrak{h})$ and $U(\mathfrak{g}) \to D(U_+) \otimes U(\mathfrak{m})$ are as in (1.9), and the right vertical isomorphism follows from the multiplication isomorphism[5] *$U_+ \times (N_+ \cap M) \simeq N_+$.*

Proof. Indeed, the following diagram commutes:

$$\begin{tikzcd} D(G)^{G_r} \arrow[r, hook] & D(G/U_-)^{M_r} \arrow[r, hook] \arrow[d] & D(G/N_-)^{H_r} \arrow[d] \\ & D(P_+)^{M_r} \arrow[r, hook] & D(P_+/(P_+ \cap N_-))^{H_r} \end{tikzcd} \tag{1.12}$$

where the vertical homomorphisms are restricting along open immersions $P_+ \subset G/U_-$ and $P_+/(P_+ \cap N_-) \subset G/N_-$. The first horizontal homomorphism $D(G)^{G_r} \hookrightarrow D(G/U_-)^{M_r}$ is obtained as follows: any $\sigma \in D(G)^{G_r}$ is an operator $\sigma\colon \mathbb{C}[G] \to \mathbb{C}[G]$ which is G_r-invariant, hence it sends $(U_-)_r$-invariant functions to $(U_-)_r$-invariant functions. In fact, for any $(U_-)_r$-invariant open subset X of

[3] the action is well-defined because M normalizes U_-.

[4] Here, as usual, letting $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution and given $M = \oplus_\mu M_\mu$, we let $M^\vee = \bigoplus_\mu M_\mu^*$ with $\langle x \cdot n, m\rangle = \langle n, -\tau(x)m\rangle$ for $n \in M^\vee, m \in M$. Alternatively, it is the *parabolic co-induction*, the right adjoint to restriction.

[5] An isomorphism of varieties; not of groups!

G, there is an operator $\sigma\colon \mathbb{C}[X]^{U_-,r} \to \mathbb{C}[X]^{U_-,r}$. In other words, since $\mathbb{C}[X/U_-] = \mathbb{C}[X]^{U_-,r}$, it defines an endomorphism of sheaves $\widetilde{\sigma}\colon \mathcal{O}_{G/U_-} \to \mathcal{O}_{G/U_-}$, which can be shown to be a differential operator. Note that we need $\widetilde{\sigma}$ to be an endomorphism of the sheaf $\mathcal{O}_{G/U_-}$, and not just $\mathbb{C}[G/U_-]$, since G/U_- may not be affine, e.g., $\mathrm{SL}_2/N_- \simeq \mathbb{A}^2\backslash\{(0,0)\}$. Moreover σ is G_r-invariant so $\widetilde{\sigma}$ must be M_r-invariant, i.e., $\widetilde{\sigma} \in D(G/U_-)^{M_r}$. All other horizontal maps are constructed in a similar fashion.

Now we have the isomorphisms $U(\mathfrak{g}) \simeq D(G)^{G_r}$ and $D(P_+)^{M_r} \simeq D(U_+) \otimes U(\mathfrak{m})$, so (1.12) can be re-written as

$$\begin{tikzcd}
U(\mathfrak{g}) \arrow[r] \arrow[dr, "\varphi^G_{P_+}"'] \arrow[rrr, bend left, "\varphi^G_{B_+}"] & D(G/U_-)^{M_r} \arrow[r] \arrow[d] & D(G/N_-)^{H_r} \arrow[r] \arrow[d] & D(N_+) \otimes U(\mathfrak{h}) \arrow[d, "\simeq"] \\
 & D(U_+) \otimes U(\mathfrak{m}) \arrow[r] \arrow[rr, bend right, "1\otimes\varphi^M_{B_+\cap M}"'] & D(P_+/(P_+ \cap N_-))^{H_r} \arrow[r] & D(U_+) \otimes D(N_+ \cap M) \otimes U(\mathfrak{h}),
\end{tikzcd}$$

which is the desired commtativity. Here the homomorphism $D(G/N_-)^{H_r} \to D(N_+) \otimes U(\mathfrak{h})$ is the composition of the restriction to the open Bruhat cell $D(G/N_-)^{H_r} \to D(B_+)^{H_r}$, together with the standard isomorphism $D(B_+)^{H_r} \simeq D(N_+) \otimes U(\mathfrak{h})$ from [Kiy24]. □

Remark 1.13. Proposition 1.11 implies Lemma 1.7.

1.2. **Back to affine Lie algebras.** Recall the definition of the Weyl algebra $\widehat{\Gamma}^{\mathfrak{g}}$ (denoted simply as $\widehat{\Gamma}$ in [Wan24b])[6]:

Definition 1.14. Let $\widehat{\Gamma}^{\mathfrak{g}} = \mathbb{C}\mathbf{1} \oplus \mathfrak{n}_+(\!(t)\!) \oplus \mathfrak{n}_+^*(\!(t)\!)dt$ with Lie bracket

$$[xf, yw] = \langle x, y\rangle \operatorname{Res}(fw) \cdot \mathbf{1} \tag{1.15}$$

for $x \in \mathfrak{n}_+$, $y \in \mathfrak{n}_+^*$, and $f \in \mathbb{C}(\!(t)\!)$, $w \in \mathbb{C}(\!(t)\!)dt$. More concretely, it has a topological basis $\mathbf{1}$, $a_{\alpha,n} := x_\alpha t^n$, and $a^*_{\alpha,n} := x^*_\alpha t^{n-1}dt$ for $\alpha \in \Delta_+$ and $n \in \mathbb{Z}$ with relations

$$[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{m+n,0}\mathbf{1} \text{ and } [a_{\alpha,n}, a_{\beta,m}] = [a^*_{\alpha,n}, a^*_{\beta,m}] = 0.$$

Let $\widehat{\Gamma}^{\mathfrak{g}}_+ = \mathfrak{n}_+[\![t]\!] \oplus \mathfrak{n}^*_+[\![t]\!]dt$, i.e., the abelian subalgebra with topological basis $a_{\alpha,n}$ for $n \geq 0$ and $a^*_{\alpha,n}$ for $n > 0$.

Given a invariant symmetric bilinear form κ on $\mathfrak{g}$, define the affine vertex algebra $V_\kappa(\mathfrak{g}) = \operatorname{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[\![t]\!]\oplus\mathbb{C}\mathbf{1}} \mathbb{C}_\kappa$ by the same formulas as in [Dum24]: for $x \in \mathfrak{g}$,

$$\mathcal{Y}(xt^{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} xt^n z^{-n-1} \tag{1.16}$$

and $[T, xt^n] = -nxt^{n-1}$. When $\mathfrak{g}$ decomposes as a direct sum, the affine vertex algebra decomposes as a tensor product:

Lemma 1.17. *Let $\mathfrak{g} = \oplus_{i=1}^s \mathfrak{g}_i \oplus \mathfrak{g}_0$ where $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ are simple Lie algebras and $\mathfrak{g}_0$ is abelian. Then there is an isomorphism*

$$V_\kappa(\mathfrak{g}) \simeq \bigotimes_{i=0}^s V_{\kappa_i}(\mathfrak{g}_i),$$

where:

[6] [Fre07, §5.3.3] denotes this Lie algebra as $\mathscr{A}^{\mathfrak{g}}$, but we avoid this notation since in [Kiy24] it denotes an associative algebra. In our notes, $\widetilde{\mathscr{A}}^{\mathfrak{g}}$ denotes an associative algebra with the same relations as $\Gamma^{\mathfrak{g}}$.

- $V_{\kappa_i}(\mathfrak{g}_i) = \mathrm{Ind}_{\mathfrak{g}_i[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_i} \mathbb{C}|0\rangle$ *is the vacuum module over* $\widehat{\mathfrak{g}}_{i,\kappa_i}$ *with the vertex algebra structure given as in* [Dum24].
- $V_{\kappa_0}(\mathfrak{g}_0) = \mathrm{Ind}_{\mathfrak{g}_0[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_0} \mathbb{C}|0\rangle$ *is the Fock representation of the Heisenberg algebra* $\widehat{\mathfrak{g}}_0$.

Recall from [Wan24b] (i.e., [Fre07, Theorem 6.2.1]) that the affine analog of the homomorphism $U(\mathfrak{g}) \to D(N_+) \otimes U(\mathfrak{h})$ constructed in [Kiy24] is a map of vertex algebras

$$w_\kappa \colon V_{\kappa+\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes V_{\kappa|_{\mathfrak{h}}}(\mathfrak{h}), \tag{1.18}$$

where $M_{\mathfrak{g}} = \mathrm{Ind}_{\widehat{\Gamma}^{\mathfrak{g}}_+\oplus\mathbb{C}\mathbf{1}}^{\widehat{\Gamma}^{\mathfrak{g}}} \mathbb{C}|0\rangle$ is the Fock representation of the Weyl algebra $\widehat{\Gamma}^{\mathfrak{g}}$ and a vertex algebra, i.e., it is generated by a vector $|0\rangle$ such that

$$a_{\alpha,n}|0\rangle = 0 \text{ for } n \geq 0,\ a^*_{\alpha,n}|0\rangle = 0 \text{ for } n > 0, \text{ and } \mathbf{1}|0\rangle = |0\rangle. \tag{1.19}$$

Later, we will use the explicit formula for w_{κ_c}, as stated in [Wan24b, §4] and [Fre07, Theorem 6.2.1]:

Theorem 1.20. *The homomorphism of vertex algebras* $w_{\kappa_c} \colon V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes V_0(\mathfrak{h})$ *is explicitly,*

$$w_{\kappa_c}(e_i(z)) = a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+} {:}P^i_\beta(\underline{a}^*(z))a_\beta(z){:} \tag{1.21}$$

$$w_{\kappa_c}(h_i(z)) = -\sum_{\beta\in\Delta_+} \beta(h_i){:}a^*_\beta(z)a_\beta(z){:} + b_i(z) \tag{1.22}$$

$$w_{\kappa_c}(f_i(z)) = \sum_{\beta\in\Delta_+} {:}Q^i_\beta(\underline{a}^*(z))a_\beta(z){:} + b_i(z)a^*_{\alpha_i}(z) + c_i\partial_z a^*_{\alpha_i}(z), \tag{1.23}$$

for some constants $c_i \in \mathbb{C}$*, where* P^i_β *and* Q^i_β *are explicit polynomials defined in* [Fre07, §5.2].

By the isomorphism $\widetilde{U}(V_\kappa(\mathfrak{g})) \simeq \widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ from [Wan24a, §2.3], the homomorphism w_κ induces a map on the completed universal enveloping algebras

$$\widetilde{U}_{\kappa+\kappa_c}(\widehat{\mathfrak{g}}) \to \widetilde{\mathscr{A}^{\mathfrak{g}}}\widehat{\otimes}\widetilde{U}_{\kappa|_{\mathfrak{h}}}(\widehat{\mathfrak{h}}).^{7} \tag{1.24}$$

We hope to generalize the homomorphism w_κ to arbitrary parabolics. Our goal is to prove the following, which is the affine analog of the homomorphism (1.9):

Theorem 1.25. *Let* κ *be an invariant symmetric bilinear form on* $\mathfrak{g}$*, and let* $\mathfrak{p} \subset \mathfrak{g}$ *be a parabolic subalgebra. Then there exists a map of vertex algebras*

$$w^{\mathfrak{p}}_\kappa \colon V_{\kappa+\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g},\mathfrak{p}} \otimes V_{\kappa|_{\mathfrak{m}}+\kappa_c(\mathfrak{m})}(\mathfrak{m}).$$

Here, $M_{\mathfrak{g},\mathfrak{p}}$ is also a Weyl vertex algebra, but for a smaller nilpotent Lie algebra than $\mathfrak{n}_+$. We small make this precise below.

Remark 1.26. When $\mathfrak{p} = \mathfrak{b}_+$, we have $w^{\mathfrak{p}}_\kappa = w_\kappa$ from (1.18).

Let us first define all the notation in the theorem statement.

Let Δ'_+ be the set of positive roots of $\mathfrak{g}$ occurring in $\mathfrak{u}_+$, or, equivalently, not occuring in $\mathfrak{p}_-$. The following is the generalization of $\widehat{\Gamma}^{\mathfrak{g}}$ to the parabolic setting:

Definition 1.27. Let $\widehat{\Gamma}^{\mathfrak{g},\mathfrak{p}} = \mathbb{C}\mathbf{1} \oplus \mathfrak{u}_+(\!(t)\!) \oplus \mathfrak{u}^*_+(\!(t)\!)dt$ with Lie bracket as in (1.15). Explicitly, it has topological basis $\mathbf{1}, a_{\alpha,n}, a^*_{\alpha,n}$ for $\alpha \in \Delta'_+$ and $n \in \mathbb{Z}$, with brackets

$$[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{m+n,0}\mathbf{1} \text{ and } [a_{\alpha,n}, a_{\beta,m}] = [a^*_{\alpha,n}, a^*_{\beta,m}] = 0.$$

There is a sub-Lie algebra $\widehat{\Gamma}^{\mathfrak{g},\mathfrak{p}}_+ := \mathfrak{u}_+[\![t]\!] \oplus \mathfrak{u}^*_+[\![t]\!]dt$. Let the *Fock representation* be $M_{\mathfrak{g},\mathfrak{p}} = \mathrm{Ind}_{\widehat{\Gamma}^{\mathfrak{g},\mathfrak{p}}_+\oplus\mathbb{C}\mathbf{1}}^{\widehat{\Gamma}^{\mathfrak{g},\mathfrak{p}}} \mathbb{C}|0\rangle$.

---

[7] Recall that $\widetilde{\mathscr{A}^{\mathfrak{g}}} := \widetilde{U}(\widehat{\Gamma}^{\mathfrak{g}})/(\mathbf{1}-1)$.

The Fock representation $M_{\mathfrak{g},\mathfrak{p}}$ can be given a vertex algebra structure by the same formula used for $M_{\mathfrak{g}}$. It is related to $M_{\mathfrak{g}}$ as follows:

Exercise 1.28. There is a vertex algebra isomorphism

$$M_{\mathfrak{g},\mathfrak{p}} \otimes M_{\mathfrak{m}} \simeq M_{\mathfrak{g}},$$

sending:

$$\begin{aligned} a_{\alpha,n}|0\rangle \otimes |0\rangle &\mapsto a_{\alpha,n}|0\rangle, \qquad & a^*_{\alpha,n}|0\rangle \otimes |0\rangle &\mapsto a^*_{\alpha,n}|0\rangle \text{ for } \alpha \in \Delta'_+, \text{ and} \\ |0\rangle \otimes a_{\beta,n}|0\rangle &\mapsto a_{\beta,n}|0\rangle, \qquad & |0\rangle \otimes a^*_{\beta,n}|0\rangle &\mapsto a^*_{\beta,n}|0\rangle \text{ for } \alpha \in \Delta_+ \backslash \Delta'_+. \end{aligned}$$

The proof of Theorem 1.25 follows the same strategy as [Fre07, Theorem 6.2.1], explained by [Wan24b], so we will not repeat it here.

Now Theorem 1.25 gives a homomorphism analogous to (1.24):

$$\widetilde{U}_{\kappa+\kappa_c}(\widehat{\mathfrak{g}}) \to \widetilde{\mathscr{A}^{\mathfrak{g},\mathfrak{p}}} \widehat{\otimes} \widetilde{U}_{\kappa|_{\mathfrak{m}}+\kappa_c(\mathfrak{m})}(\widehat{\mathfrak{m}}),$$

which allows us to define generalized Wakimoto modules:

Definition 1.29. Let R be a smooth $\widehat{\mathfrak{m}}_{\kappa|_{\mathfrak{m}}+\kappa_c(\mathfrak{m})}$-module. Then $M_{\mathfrak{g},\mathfrak{p}} \otimes R$ carries a smooth $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$-module structure, called the *generalized Wakimoto module corresponding to* R. We denote it by $\mathrm{Wak}^{\mathfrak{g}}_{\mathfrak{p}} R$.

Now we have the following analog of Lemma 1.7, which finally accomplishes Goal 1.5 (see [Los24b] for a proof sketch):

Proposition 1.30. *There is a commutative diagram:*

$$\begin{array}{ccc} V_{\kappa+\kappa_c}(\mathfrak{g}) & \xrightarrow{\quad w_\kappa \quad} & M_{\mathfrak{g}} \otimes V_{\kappa|_{\mathfrak{h}}}(\mathfrak{h}) \\ \big\downarrow{\scriptstyle w^{\mathfrak{p}}_\kappa} & & \big\downarrow{\scriptstyle \simeq} \\ M_{\mathfrak{g},\mathfrak{p}} \otimes V_{\kappa|_{\mathfrak{m}}+\kappa_c(\mathfrak{m})}(\mathfrak{m}) & \xrightarrow{1\otimes w_{\kappa|_{\mathfrak{m}}}} & M_{\mathfrak{g},\mathfrak{p}} \otimes M_{\mathfrak{m}} \otimes V_{\kappa|_{\mathfrak{h}}}(\mathfrak{h}), \end{array}$$

where the vertical isomorphism was defined in Exercise 1.28. Thus, for any $\lambda \in \mathfrak{h}^*$ there is an isomorphism

$$\mathrm{Wak}^{\mathfrak{g}}_{\mathfrak{p}}(W_{\lambda,\kappa|_{\mathfrak{m}}+\kappa_c(\mathfrak{m})}) \simeq W_{\lambda,\kappa+\kappa_c}.$$

2. Comparing affine Verma modules to Wakimoto modules

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra now. Let $\widetilde{\mathfrak{b}}_+ := \mathfrak{b}_+ + t\mathfrak{g}[\![t]\!]$ and $\widetilde{\mathfrak{n}}_+ := \mathfrak{n}_+ + t\mathfrak{g}[\![t]\!]$ be the pre-images of $\mathfrak{b}_+$ and $\mathfrak{n}_+$, respectively, under the quotient map $\mathfrak{g}[\![t]\!] \to \mathfrak{g}$ evaluating at $t = 0$. The subalgebra $\widetilde{\mathfrak{b}}_+$ is called the *Iwahori subalgebra*, and $\widetilde{\mathfrak{n}}_+$ is its topological nilpotent radical. Now for a weight $\lambda \in \mathfrak{h}^*$ let $\mathbb{C}_\lambda$ be the one-dimensional representation of $\widetilde{\mathfrak{b}}_+ \oplus \mathbb{C}\mathbf{1}$ such that $\widehat{\mathfrak{n}}_+$ acts by zero, $\mathfrak{h}$ acts by λ, and $\mathbf{1}$ acts as the identity.

Definition 2.1. The *Verma module* $\mathbb{M}_{\lambda,\kappa}$ of level κ and highest weight λ is

$$\mathbb{M}_{\lambda,\kappa} := \mathrm{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\widetilde{\mathfrak{b}}_+ \oplus \mathbb{C}\mathbf{1}} \mathbb{C}_\lambda.$$

Denote the highest-weight vector, $1 \otimes 1$, as $v_{\lambda,\kappa}$.

We hope to compare the Wakimoto module W_{0,κ_c} with the Verma module $\mathbb{M}_{0,\kappa_c}$. There is a homomorphism

$$\mathbb{M}_{0,\kappa_c} \twoheadrightarrow V_{\kappa_c}(\mathfrak{g}) \xrightarrow{w_{\kappa_c}} W_{0,\kappa_c} \tag{2.2}$$

which sends the highest-weight vector v_{0,κ_c} to $|0\rangle \otimes |0\rangle$, since by construction w_{κ_c} is $\widehat{\mathfrak{g}}_{\kappa_c}$-equivariant. Here, the first homomorphism is by the transitivity of induction:

$$\mathbb{M}_{0,\kappa_c} = \operatorname{Ind}_{\widetilde{\mathfrak{b}}_+}^{\widehat{\mathfrak{g}}} \mathbb{C}_0 = \operatorname{Ind}_{\mathfrak{g}[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}} \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_0 \twoheadrightarrow \operatorname{Ind}_{\mathfrak{g}[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa_c}} \mathbb{C} = V_{\kappa_c}(\mathfrak{g}).$$

However, (2.2) cannot be an isomorphism; indeed, the energy zero component of $\mathbb{M}_{\lambda,\kappa_c}$ is the Verma module $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_0$ while the energy zero component of W_{0,κ_c} is the dual Verma module $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_0)^\vee$, so they cannot be isomorphic. Thus, we modify the Wakimoto modules $W_{\lambda,\kappa}$ to $W^+_{\lambda,\kappa}$ to be defined below, so that the following holds:

Theorem 2.3 ([Fre07, Proposition 6.3.3])**.** *The Wakimoto module W^+_{0,κ_c} is isomorphic to the Verma module $\mathbb{M}_{0,\kappa_c}$.*

To define $W^+_{\lambda,\kappa}$, the Fock representation of $\widehat{\Gamma}^{\mathfrak{g}}$, defined as $M_{\mathfrak{g}} := \operatorname{Ind}_{\widehat{\Gamma}^{\mathfrak{g}}_+\oplus\mathbb{C}\mathbf{1}}^{\widehat{\Gamma}^{\mathfrak{g}}} \mathbb{C}|0\rangle$, is modified to the module with the following modification of (1.19):

$$a_{\alpha,n}|0\rangle' = 0 \text{ for } n > 0,\ a^*_{\alpha,n}|0\rangle' = 0 \text{ for } n \geq 0, \text{ and } \mathbf{1}|0\rangle' = |0\rangle'.$$

Now let

$$W^+_{\lambda,\kappa} := M'_{\mathfrak{g}} \otimes \pi^{\kappa-\kappa_c}_{-2\rho-\lambda}, \tag{2.4}$$

where $\pi^{\kappa-\kappa_c}_{-2\rho-\lambda}$ was defined in [Wan24b, §0]:

$$\pi^{\kappa-\kappa_c}_{-2\rho-\lambda} := \operatorname{Ind}_{\mathfrak{h}[\![t]\!]\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{h}}_{\kappa-\kappa_c}} \mathbb{C}|-2\rho-\lambda\rangle.$$

Denote the vector $|0\rangle' \otimes |-2\rho-\lambda\rangle$ in $W^+_{\lambda,\kappa}$ as $|0\rangle'$. The shift by 2ρ in (2.4) is explained in §2.2; it is necessary for $|0\rangle' \in W^+_{\lambda,\kappa}$ to be a highest weight vector of weight λ.

We may modify the formulas in Theorem 1.20 to obtain a homomorphism $\widehat{\mathfrak{g}}_\kappa$-module structure on $W^+_{\lambda,\kappa}$. We will give explicit formulas at the critical level:

Theorem 2.5. *The module $W^+_{\lambda,\kappa}$ has a $\widehat{\mathfrak{g}}_{\kappa_c}$-module structure given by*

$$w'_{\kappa_c}(f_i(z)) = a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+} {:}P^i_\beta(\underline{a}^*(z))a_\beta(z){:} \tag{2.6}$$

$$w'_{\kappa_c}(h_i(z)) = \sum_{\beta\in\Delta_+} \beta(h_i){:}a^*_\beta(z)a_\beta(z){:} - b_i(z) \tag{2.7}$$

$$w'_{\kappa_c}(e_i(z)) = \sum_{\beta\in\Delta_+} {:}Q^i_\beta(\underline{a}^*(z))a_\beta(z){:} + b_i(z)a^*_{\alpha_i}(z) + c_i\partial_z a^*_{\alpha_i}(z), \tag{2.8}$$

for some constants $c_i \in \mathbb{C}$ and where polynomials P^i_β and Q^i_β are defined in [Fre07, §5.2].

In fact, there are formulas for $w'_{\kappa_c}(f_\alpha(z))$ for arbitrary $\alpha \in \Delta_+$, not just for simple roots:

$$w'_{\kappa_c}(f_\alpha(z)) = a_\alpha(z) + \sum_{\beta\in\Delta_+;\beta>\alpha} {:}P^\alpha_\beta(\underline{a}^*(z))a_\beta(z){:} \tag{2.9}$$

for some polynomials P^α_β. See [Fre07, equation (6.1-2)].

We prove Theorem 2.3 in three steps:

(a) comparing the formal characters;
(b) constructing a homomorphism $\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c}$; and
(c) proving the surjectivity of the homomorphism.

From the three steps, the isomorphism is clear: the character of the kernel of $\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c}$ must be zero by (a). Step (b) is accomplished in exactly the same way the homomorphism $\mathbb{M}_{0,\kappa_c} \to W_{0,\kappa_c}$ was constructed in (2.2), by sending the highest vector v_{0,κ_c} to the vacuum vector $|0\rangle' \otimes |-2\rho\rangle$.

2.1. **Formal characters of $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$-modules.** To check (a), let us recall what the character of a $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$-module is. For a $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$-module M, suppose there is a grading operator $d\colon M \to M$ compatible with the $\widehat{\mathfrak{g}}_\kappa$-action, i.e., such that $[d, xt^n] = nxt^{n-1}$. Let $\mathfrak{h}' = \mathfrak{h} \oplus \mathbb{C}d$, so the characters are of the form $\lambda' = (\lambda, \phi)$ where $\lambda \in \mathfrak{h}^*$ and $\phi \in \mathbb{C}$, so d acts by ϕ.

Now, we can define the character of a $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$-module:

Definition 2.10. Let M be a smooth $\widehat{\mathfrak{g}}'_\kappa$-module, such that $\mathbf{1}$ acts by identity and the Cartan $\mathfrak{h} \oplus \mathbb{C}d \oplus \mathbb{C}\mathbf{1}$ acts semi-simply on M with finite-dimensional weight spaces:

$$M = \bigoplus_{\lambda' \in (\mathfrak{h}')^*} M(\lambda').$$

Then the *character* of M is

$$\operatorname{ch} M = \sum_{\lambda' \in (\mathfrak{h}')^*} \dim M(\lambda') \cdot e^{\lambda'}.$$

Letting $\delta := (0, 1) \in \widetilde{\mathfrak{h}}^*$, the set of positive roots of $\widehat{\mathfrak{g}}'$ is:

$$\widehat{\Delta}_+ = (\Delta_+ + \mathbb{Z}_{\geq 0}\delta) \sqcup \big((\Delta_- \cup \{0\}) + \mathbb{Z}_{>0}\delta\big). \tag{2.11}$$

The positive roots define a partial order on $\widetilde{\mathfrak{h}}^*$:

Definition 2.12. Let $\lambda' > \mu'$ if $\lambda' - \mu' = \sum_i \beta'_i$ for some $\beta'_i \in \widehat{\Delta}_+$.

The Verma module $\mathbb{M}_{\lambda,\kappa}$ over $\widehat{\mathfrak{g}}_\kappa$, as defined in Definition 2.1, can be extended to $\widehat{\mathfrak{g}}'_\kappa$, which we denote by $\mathbb{M}_{\lambda',\kappa}$ where $\lambda' = (\lambda, 0)$:

$$\mathbb{M}_{\lambda',\kappa} := \operatorname{Ind}_{\widetilde{\mathfrak{b}}_+ \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}d}^{\widehat{\mathfrak{g}}'_\kappa} \mathbb{C}_{\lambda'}.$$

Now by the PBW theorem, as a vector space $\mathbb{M}_{\lambda,\kappa} \simeq U(\widetilde{\mathfrak{n}}_-)$, where $\widetilde{\mathfrak{n}}_- = \mathfrak{n}_- \oplus t^{-1}\mathfrak{g}[t^{-1}]$, so

$$\operatorname{ch} \mathbb{M}_{\lambda',\kappa} = e^{\lambda'} \prod_{\alpha' \in \widehat{\Delta}_+} (1 - e^{-\alpha'})^{-\operatorname{mult} \alpha'}, \tag{2.13}$$

where $\operatorname{mult} \alpha'$ is the dimension of the weight space $\widehat{\mathfrak{g}}'_{\kappa_c,\alpha'}$.

Since W^+_{0,κ_c} has a basis in the monomials

$$a_{\alpha,n}, \alpha \in \Delta_+, n < 0; a^*_{\alpha,n}, \alpha \in \Delta_+, n \leq 0; \text{ and } b_{i,n}, i = 1, \ldots, \ell, n < 0, \tag{2.14}$$

to compute the character of the $\widehat{\mathfrak{g}}'_\kappa$-module W^+_{0,κ_c}, we must compute the $\mathfrak{h}'$-action on $a_{\alpha,n}$, $a^*_{\alpha,n}$, and $b_{i,n}$.

Since d simply acts by $L_0 = -t\partial_t$ on $M_\mathfrak{g} \otimes V_0(\mathfrak{h})$,

$$[d, a_{\alpha,n}] = -na_{\alpha,n}, \quad [d, a^*_{\alpha,n}] = -na^*_{\alpha,n}, \quad [d, b_{i,n}] = -nb_{i,n}, \tag{2.15}$$

where $a_{\alpha,n}, a^*_{\alpha,n} \in M_\mathfrak{g}$, and $b_{i,n} \in \widetilde{U}_0\big(\mathfrak{h}^*(\!(t)\!)\big)$. The $\mathfrak{h}$-action on W^+_{0,κ_c} is given by, for $h \in \mathfrak{h}$,

$$[h, a_{\alpha,n}] = \alpha(h)a_{\alpha,n}, \quad [h, a^*_{\alpha,n}] = \alpha(h)a^*_{\alpha,n}, \quad [h, b_{i,n}] = 0. \tag{2.16}$$

Formula (2.16) follows from (1.22):

Exercise 2.17. Deduce formula (2.16) from (1.22).

Now (2.15) and (2.16) together show that the character of W^+_{0,κ_c} equals (2.13).

2.2. **Constructing the homomorphism.** Let us compute the action of $\mathfrak{h}$ on $|0\rangle'$:

Exercise 2.18. For any $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$, then $h \cdot |0\rangle' = \lambda(h)|0\rangle'$ in W^+_{0,κ_c}.

The Exercise shows why the shift by 2ρ was necessary in (2.4). The classical analog is the following: $\mathbb{C}[x]$ and $\mathbb{C}[\delta_0]$ are both $D(\mathbb{A}^1) \simeq \mathbb{C}[x, \partial_x]$-modules, where δ_0 is the delta function supported on 0.[8] Then $L_0 = -x\partial_x$ acts as 0 on $1 \in \mathbb{C}[x]$, but acts instead as

$$-x\partial_x \cdot 1 = (1 - \partial_x x)1 = 1.$$

Solution to Exercise 2.18. The constant term in (2.7) is (by definition of the normally ordered product)

$$\begin{aligned}
w_{\kappa_c}(h_{i,0}|0\rangle) &= \sum_{\beta\in\Delta_+} \beta(h_i)\Big(\sum_{n\geq 0} a^*_{\beta,-n}a_{\beta,n} + \sum_{n<0} a_{\beta,n}a^*_{\beta,-n}\Big)|0\rangle' - b_{i,0}|0\rangle' \\
&= \sum_{\beta\in\Delta_+} \beta(h_i)a^*_{\beta,0}a_{\beta,0}|0\rangle' - (-2\rho - \lambda)(h_i)|0\rangle' \\
&= \sum_{\beta\in\Delta_+} \beta(h_i)(a_{\beta,0}a^*_{\beta,0} - 1)|0\rangle' + (2\rho + \lambda)(h_i)|0\rangle' \\
&= -\sum_{\beta\in\Delta_+} \beta(h_i)|0\rangle' + (2\rho + \lambda)(h_i)|0\rangle' \\
&= \lambda(h_i)|0\rangle'.
\end{aligned}$$ □

Now, by the character formula in §2.1 the weight spaces of $\lambda' > 0$ are zero, i.e., $|0\rangle' \in W^+_{0,\kappa_c}$ is annihilated by $\widetilde{\mathfrak{n}}_+$. Thus there is a homomorphism $\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c}$.

2.3. **Proving the surjectivity of the homomorphism.**

The remainder of the proof of Theorem 2.3. We need to check that $\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c}$ is surjective, i.e., that W^+_{0,κ_c} is generated as a $\widehat{\mathfrak{g}}_{\kappa_c}$-module by $|0\rangle'$. Consider the coinvariants of W^+_{0,κ_c} with respect to $\widetilde{\mathfrak{n}}_- = \mathfrak{n}_- \oplus t^{-1}\mathfrak{g}[t^{-1}]$:

$$(W^+_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-} := \mathbb{C}_0 \otimes_{U(\widetilde{\mathfrak{n}}_-)} W^+_{0,\kappa_c},$$

which is a $\mathfrak{h}'$-representation since $\widetilde{\mathfrak{n}}_- \subset \widehat{\mathfrak{g}}_{\kappa_c}$ is $\mathfrak{h}'$-stable. If $\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c}$ were not surjective, then there is an exact sequence of $\widehat{\mathfrak{g}}'_{\kappa_c}$-modules

$$\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c} \to V \to 0,$$

for some non-zero V, which induces an exact sequence of $\mathfrak{h}'$-modules

$$(\mathbb{M}_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-} = \mathbb{C} \to (W^+_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-} \to V_{\widetilde{\mathfrak{n}}_-} \to 0, \tag{2.19}$$

where $V_{\widetilde{\mathfrak{n}}_-} \neq 0$. But $(\mathbb{M}_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-} \to (W^+_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-}$ is an isomorphism on the $(0,0)$-weight space, so $(W^+_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-}$ must have a nonzero weight μ'. In other words W^+_{0,κ_c} has an irreducible quotient L_{μ',κ_c} with highest weight μ'. Since $\mathbb{M}_{0,\kappa_c}$ and W^+_{0,κ_c} have the same characters, they define the same class in the Grothendieck group and hence must have the same irreducible subquotients. Now we shall:

(1) observe restrictions on $\mu' \in (\mathfrak{h}')^*$ coming from L_{μ',κ_c} being a subquotient of W^+_{0,κ_c}; and
(2) observe restrictions on $\mu' \in (\mathfrak{h}')^*$ coming from L_{μ',κ_c} being a subquotient of $\mathbb{M}_{0,\kappa_c}$.

[8] They are Fourier transforms of each other.

We will show the two restrictions on μ' are incompatible, and hence our assumption, that $V \neq 0$, must have been wrong.

First, however, *there is a sublety*: $\mathbb{M}_{0,\kappa_c}$ and W^+_{0,κ_c} have infinite length, so $\operatorname{ch}\mathbb{M}_{0,\kappa_c} = \operatorname{ch} W^+_{0,\kappa_c}$ does *not* imply they have the same irreducible subquotients in the naïve way. The correct statement is as follows:

Exercise 2.20. Let M and N be category $\mathcal{O}$-modules for $\widehat{\mathfrak{g}}'_\kappa$. Then $\operatorname{ch} M = \operatorname{ch} N$ if and only if M and N define the same class in the *completed* Grothendieck group $\widehat{K}_0(\mathcal{O}_{\widehat{\mathfrak{g}}'_\kappa})$, which is the inverse limit

$$\widehat{K}_0(\mathcal{O}_{\widehat{\mathfrak{g}}'_\kappa}) := \varprojlim_{\lambda' \in (\mathfrak{h}')^*} K_0(\mathcal{O}_{\widehat{\mathfrak{g}}'_\kappa}/\mathcal{O}_{\widehat{\mathfrak{g}}'_\kappa, \leq \lambda'}),$$

over the partial order on $(\mathfrak{h}')^*$ defined in (2.12) where $\mathcal{O}_{\widehat{\mathfrak{g}}'_\kappa, \leq \lambda'}$ is the Serre subcategory of $\mathcal{O}_{\widehat{\mathfrak{g}}'_\kappa}$ consisting of modules with weights $\leq \lambda'$. Moreover when this holds, if L is an irreducible subquotient of M, then L is also an irreducible subquotient of N.

For (1), note that by the explicit formulas for $f_\alpha(z)$ and $h_i(z)$-actions on W^+_{0,κ_c} in (2.7) and (2.9), respectively, the lexicographically ordered monomials

$$\prod_{\ell_\alpha < 0} b_{i_\alpha, \ell_\alpha} \prod_{m_b \leq 0} f_{\alpha_b, m_b} \prod_{n_c < 0} a^*_{\beta_c, n_c} |0\rangle' \text{ where } 1 \leq i_\alpha \leq \ell,\ \alpha_b \in \Delta_s, \text{ and } \beta_c \in \Delta_+ \tag{2.21}$$

form a basis of W^+_{0,κ_c}. The weights appearing in the coinvariants must be of the form

$$\mu' = -\sum_j (n_j \delta - \beta_j) \tag{2.22}$$

where $n_j > 0$ and $\beta_j \in \Delta_+$. Indeed, by the description of the basis of W^+_{0,κ_c} in (2.21), there is an isomorphism of $\widetilde{\mathfrak{h}}$-modules

$$(W^+_{0,\kappa_c})_{\mathfrak{n}_-[t^{-1}] \oplus t^{-1}\mathfrak{h}[t^{-1}]} \simeq \mathbb{C}[a^*_{\alpha,n}]_{\alpha \in \Delta_+, n<0},$$

and $(W^+_{0,\kappa_c})_{\widetilde{\mathfrak{n}}_-}$ is a quotient.

For (2) note that [KK79, Theorem 2] (also see [Fre07, §6.3.3]) gives a characterization of possible irreducible subquotient of Verma modules:

Proposition 2.23. *A weight $\mu' = (\mu, n)$ appears as the highest weight of an irreducible subquotient of $\mathbb{M}_{(\lambda,0),\kappa_c}$ if and only if $n \leq 0$ and $\mu = w(\rho) - \rho$ for some $w \in W$.*

Note that for any $w \in W$ the weight $w(\rho) - \rho$ equals the linear combination of simple roots of $\mathfrak{g}$ with non-positive coefficients, hence the weight of any irreducible subquotient of $\mathbb{M}_{0,\kappa_c}$ has the form

$$\mu' = -n\delta - \sum_i m_i \alpha_i \tag{2.24}$$

for some $n \geq 0$ and $m_i \geq 0$. Finally, note that (2.22) and (2.24) cannot simultaneously hold, a contradiction, and hence $V = 0$. We have thus completed (a), (b), and (c), which together prove that $\mathbb{M}_{0,\kappa_c} \simeq W^+_{0,\kappa_c}$. □

Next, we characterize all the endomorphisms of our module $\mathbb{M}_{0,\kappa_c} \simeq W^+_{0,\kappa_c}$. In other words, we hope to characterize all $\widehat{\mathfrak{g}}'_{\kappa_c}$-homomorphisms $\mathbb{M}_{0,\kappa_c} \to W^+_{0,\kappa_c}$. By adjunction, this is equivalent to characterize the vectors in W^+_{0,κ_c} annihilated by $\widetilde{\mathfrak{b}}_+$.

Lemma 2.25 ([Fre07, Lemma 6.3.4])**.** *The space of $\widetilde{\mathfrak{b}}_+$-invariants of W^+_{0,κ_c} is equal to $\pi_{-2\rho} \subset W^+_{0,\kappa_c}$.*

Proof. The formulas in Theorem 2.5 shows the vectors of $\pi_{-2\rho}$ are annihilated by $\widetilde{\mathfrak{b}}_+$. To prove the converse, note that W^+_{0,κ_c} has another basis

$$\prod_{\ell_\alpha<0} b_{i_\alpha,\ell_\alpha} \prod_{m_b\leq 0} f^R_{\alpha_b,m_b} \prod_{n_c<0} a^*_{\alpha_c,n_c}|0\rangle',$$

by the same argument as for (2.21). Here the $f^R_{\alpha,n}$ generate an action of $t\mathfrak{n}_-[\![t]\!]$ as defined in [Los24a], which we now briefly recall. There is an isomorphism of the Fock representation of $\widehat{\Gamma}^{\mathfrak{g}}$ with the vertex algebra of chiral differential operators on N_-:

$$M_{\mathfrak{g}} \simeq \mathrm{CDO}(N_-).$$

Now viewing $\mathcal{J}\mathfrak{n}_- = \mathfrak{n}_-[\![t]\!]$ as the *right-invariant* vector fields on $\mathcal{J}N_-$ defines a left $\mathfrak{n}_-[\![t]\!]$-action on $\mathrm{CDO}(N_-)$, which induces the restriction of the $\widehat{\mathfrak{g}}'_{\kappa_c}$-action on W^+_{0,κ_c}. On the other hand, viewing $\mathfrak{n}_-[\![t]\!]$ as the *left-invariant* vector fields on $\mathcal{J}N_-$ defines a right $\mathfrak{n}_-[\![t]\!]$-action which are the $f^R_{\alpha,n}$.

Thus there is a tensor product decomposition

$$W^+_{0,\kappa_c} = \overline{W}^+_{0,\kappa_c} \otimes W^{+,*}_{0,\kappa_c},$$

where $W^{+,*}_{0,\kappa_c}$ (resp., $\overline{W}^+_{0,\kappa_c}$) is the span of monomials in $a^*_{\alpha,n}$ (resp., in $b_{i,\ell}$ and $f^R_{\alpha.m}$). Since the left action of $t\mathfrak{n}_-[\![t]\!]$ commutes with $b_{i,\ell}$ and $f^R_{\alpha,m}$, we conclude $t\mathfrak{n}_-[\![t]\!]$ acts by zero on $\overline{W}^+_{0,\kappa_c}$. In fact, it is isomorphic to the restricted dual of the free $\widetilde{U}(\mathfrak{n}_-[\![t]\!])$-module with one generator. Thus

$$(W^+_{0,\kappa_c})^{t\mathfrak{n}_-[\![t]\!]} = \overline{W}^+_{0,\kappa_c} \otimes (W^{+,*}_{0,\kappa_c})^{t\mathfrak{n}_-[\![t]\!]} = \overline{W}^+_{0,\kappa_c}.$$

Furthermore, for $h \in \mathfrak{h}$ since

$$[h, a^*_{\alpha,n}] = \alpha(h)a^*_{\alpha,n},$$

a vector in $\overline{W}^+_{0,\kappa_c}$ is annihilated by $\mathfrak{h}$ if and only if it belongs to $\pi_{-2\rho}$. □

3. Proof of the Kac-Kazhdan conjecture

The Verma module $\mathbb{M}_{\lambda',\kappa}$ over $\widehat{\mathfrak{g}}'_\kappa$ has a unique irreducible quotient $L_{\lambda',\kappa}$. The Kac-Kazhdan conjecture computes the character of $\mathbb{M}_{\lambda',\kappa}$ for generic λ'.

First, recall that the roots $\widehat{\Delta}_+$ from (2.11) has a subset of *real roots*

$$\widehat{\Delta}^{\mathrm{re}}_+ := (\Delta_+ + \mathbb{Z}_{\geq 0}\delta) \sqcup (\Delta_- + \mathbb{Z}_{>0}\delta),$$

i.e., the roots $(\lambda,\phi) \in \widehat{\Delta}_+$ such that $\lambda \neq 0$.

Theorem 3.1. *For a generic weight $\lambda \in \mathfrak{h}^*$ of critical level,*

$$\mathrm{ch}\, L_{\lambda',\kappa_c} = e^{\lambda'} \prod_{\alpha' \in \widehat{\Delta}^{\mathrm{re}}_+} (1 - e^{-\alpha'})^{-1}.$$

Here, a weight λ is generic *when $\lambda \notin \bigcup_{\beta \in \widehat{\Delta}^{\mathrm{re}}_+, m>0} H^{\kappa_c}_{\beta,m}$ where $H^{\kappa_c}_{\beta,m}$ are certain hyperplanes in $\mathfrak{h}^*$ defined in* [KK79].

Proof. For $\lambda \in \mathfrak{h}^*$ the Wakimoto module of critical level $W_{\lambda/t}$ is a $\widehat{\mathfrak{g}}'_{\kappa_c}$-module since $M_{\mathfrak{g}}$ is graded, and the $\mathfrak{h}(\!(t)\!)$-module $\mathbb{C}_{\lambda/t}$ is graded, and hence $W_{\lambda/t} := M_{\mathfrak{g}} \otimes \mathbb{C}_{\lambda/t}$ inherits a grading. The $\widehat{\mathfrak{g}}'_{\kappa_c}$-module $W_{\lambda/t}$ has character

$$\mathrm{ch}\, W_{\lambda/t} = e^{\lambda'} \prod_{\alpha' \in \widehat{\Delta}^{\mathrm{re}}_+} (1 - e^{-\alpha'})^{-1},$$

where $\lambda' = (\lambda, 0)$, from a similar argument as in (a) in the proof of Theorem 2.3. Moreover, the same argument as in (b) in the proof of Theorem 2.3 shows there is a homomorphism $\mathbb{M}_{\lambda',\kappa_c} \to W_{\lambda/t}$

sending the highest weight vector to $|0\rangle$. It thus suffices to check that if λ is a generic weight of critical level, then $W_{\lambda/t}$ is irreducible, since then $L_{\lambda',\kappa_c} \simeq W_{\lambda/t}$. If $W_{\lambda/t}$ is not irreducible, either:

- $W_{\lambda/t}$ is not generated by its highest vector, i.e., the homomorphism $\mathbb{M}_{\lambda',\kappa_c} \to W_{\lambda/t}$ is not surjective; or
- $W_{\lambda/t}$ is generated by its highest vector, in which $\mathbb{M}_{\lambda',\kappa_c} \to W_{\lambda/t}$ is surjective and the image of a highest weight of the maximal sub-module of $\mathbb{M}_{\lambda',\kappa_c}$ is a non-zero singular vector in $W_{\lambda/t}$ not in $\mathbb{C}|0\rangle$.

If $W_{\lambda/t}$ contains a singular vector not in $\mathbb{C}|0\rangle$ then it must be annihilated by $\mathfrak{n}_+[\![t]\!]$. We know that

$$\prod_{n_a<0} e^R_{\alpha_a,n_a} \prod_{m_b\leq 0} a^*_{\alpha_b,m_b}|0\rangle$$

forms a basis of $M_{\mathfrak{g}}$, where the $e^R_{\alpha_a,n_a}$ are defined as in the proof of Lemma 2.25, using the description of $M_{\mathfrak{g}} \simeq \mathrm{CDO}(N_+)$, as in the proof of Lemma 2.25. By the same method as in Lemma 2.25, the $\mathfrak{n}_+[\![t]\!]$-invariants of W_{0,κ_c} equals the subspace $\overline{W}_{0,\kappa_c}$ spanned by all monomials of $e^R_{\alpha_a,n_a}$. In particular, the weight of any singular vector of $W_{\lambda/t}$ is of the form $\lambda' - \sum_j (n_j\delta - \beta_j)$ where $n_j > 0$ and $\beta_j \in \Delta_+$. Thus $W_{\lambda/t}$ contains an irreducible subquotient of that highest weight. Now, since for $\alpha' \in \widehat{\Delta}_+$,

$$\operatorname{mult}\alpha' = \begin{cases} 1 & \text{if } \alpha' \in \widehat{\Delta}^{\mathrm{re}}_+ \\ \ell & \text{otherwise,} \end{cases}$$

we have

$$\operatorname{ch}\mathbb{M}_{\lambda',\kappa_c} = \prod_{n>0}(1-e^{-n\delta})^{-\ell}\operatorname{ch}W_{\lambda/t}, \tag{3.2}$$

where ℓ is the rank of $\mathfrak{g}$. Thus if an irreducible module L_{μ',κ_c} appears as a subquotient of $W_{\lambda/t}$, it must also appear as a subquotient of $\operatorname{ch}\mathbb{M}_{\lambda',\kappa_c}$: only look at the part of (3.2) with energy zero. But our contradicts the assumption that λ is generic: irreducible subquotients of Verma modules are controlled by hyperplanes by [KK79]. Thus $W_{\lambda/t}$ does not contain any singular vectors other than the highest weight.

Next, if $W_{\lambda/t}$ is not generated by its highest vector, then by the same argument as above there is an irreducible subquotient of $W_{\lambda/t}$ with highest weight $\lambda' - \sum_j (n_j\delta + \beta_j)$ with $n_j \geq 0$ and $\beta_j \in \Delta_+$. This again contradicts λ being a generic weight. □

References

[Dum24] Ilya Dumanski, *Vertex algebras*, seminar notes 2024.

[Fre07] Edward Frenkel, *Langlands correspondence for loop groups*, Cambridge Studies in Advanced Mathematics, vol. 103, Cambridge University Press, Cambridge, 2007. MR 2332156

[Kiy24] Daishi Kiyohara, *Free field realization*, seminar notes 2024.

[KK79] V. G. Kac and D. A. Kazhdan, *Structure of representations with highest weight of infinite-dimensional Lie algebras*, Adv. in Math. **34** (1979), no. 1, 97–108. MR 547842

[KL24] Ivan Karpov and Ivan Losev, *Invariants of jets and the center for* $\widehat{\mathfrak{sl}}_2$, seminar notes 2024.

[Los24a] Ivan Losev, *Chiral differential operators 1*, seminar notes 2024.

[Los24b] ———, *Chiral differential operators 2*, seminar notes 2024.

[Wan24a] Hamilton Wan, *Central elements of the completed universal enveloping algebra*, seminar notes 2024.

[Wan24b] Zeyu Wang, *Wakimoto modules*, seminar notes 2024.

M.I.T., 77 Massachusetts Avenue, Cambridge, MA, USA

Email address: kjsuzuki@mit.edu