We will define generalized Wakimoto modules, which gives a functorial way of constructing \( \widehat{g} \)-modules from \( \widehat{\mathfrak{m}} \)-modules for parabolic subalgebras \( p = m \ltimes u \subset g \). We will give applications of Wakimoto modules, including the Kac-Kazhdan conjecture, which computes the characters of Verma modules \( M_{\lambda, \kappa} \) on the critical level for \( \lambda \in \mathfrak{h}^* \) generic, i.e., not lying in a countable union of hyperplanes.

1. Semi-infinite parabolic induction

Let \( g \) be a finite-dimensional reductive Lie algebra with Borel subalgebra \( \mathfrak{b}_+ \) and Cartan subalgebra \( \mathfrak{h} \) (we work in this generality since we want to apply our construction to Levi subalgebras of simple Lie algebras).

Wakimoto modules, constructed in [Wan24b], are the images of Fock modules under a functor \( \mathcal{F}_{\kappa}(\mathfrak{h}) \text{-mod} \to \mathcal{F}_{\kappa, \mathfrak{h}}(\mathfrak{g}) \text{-mod}. \) We want to generalize the construction by replacing the Borel subalgebra \( \mathfrak{b} \) with an arbitrary parabolic subalgebra \( p \) and replacing the Cartan subalgebra \( \mathfrak{h} \) with the Levi component \( m \) of \( p \). Let us first recall what a parabolic subalgebra is:

**Definition 1.1.** A parabolic subalgebra is a subalgebra \( p \subset g \) such that one of the following equivalent conditions hold:

- \( p \) contains a Borel subalgebra of \( g \); or
- the orthogonal complement of \( p \) with respect to an invariant orthogonal form\(^2\) is its nilradical.

**Example 1.2.** \( \mathfrak{b}_+ \) and \( g \) are parabolic subalgebras of \( g \).

Each conjugacy class of parabolic subalgebras has a unique representative containing \( \mathfrak{b}_+ \): we call those parabolic subalgebras standard. Let \( \Delta_s \) be the set of simple roots corresponding to \( \mathfrak{b}_+ \subset g \). Then standard parabolic subalgebras of \( g \) are classified by subsets of \( \Delta_s \): so \( \mathfrak{b}_+ \) corresponds to \( \emptyset \) and \( g \) corresponds to \( \Delta_s \). More generally, for a subset \( S \subset \Delta_s \), the corresponding standard parabolic subalgebra \( p_S \subset g \) is

\[
p_S := \mathfrak{b}_+ \oplus \bigoplus_{\alpha > 0, \alpha \in \text{span} \Delta_s} \mathfrak{g}_\alpha.
\]

The Levi component is then given by:

\[
m_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in \text{span} \Delta_s} \mathfrak{g}_\alpha.
\]

Analogous to the opposite Borel subalgebra, let

\[
p_{S, -} := \mathfrak{b}_- \oplus \bigoplus_{\alpha < 0, \alpha \in \text{span} \Delta_s} \mathfrak{g}_\alpha
\]

be the opposite parabolic.\(^1\)

---

\(^1\)These are categories of smooth modules.

\(^2\)When \( g \) is semisimple we can use the Killing form, but for arbitrary reductive Lie algebras the Killing form may be degenerate.
**Example 1.3.** When \( g = \mathfrak{sl}_n \), let \( S \) be a subset of \( \Delta_s = \{\alpha_1, \ldots, \alpha_{n-1}\} \) such that \( \Delta_s \setminus S = \{\alpha_1, \ldots, \alpha_k\} \). The corresponding parabolic subalgebras are

\[
p_S = \mathfrak{sl}_n \cap \begin{pmatrix} M_{a_1 \times a_1} & * & * & * \\
0 & M_{(a_2-a_1) \times (a_2-a_1)} & * & * \\
0 & 0 & \ddots & * \\
0 & 0 & 0 & M_{(n-a_k) \times (n-a_k)} \end{pmatrix}
\]

and

\[
p_{S,-} = \mathfrak{sl}_n \cap \begin{pmatrix} M_{a_1 \times a_1} & * & M_{(a_2-a_1) \times (a_2-a_1)} & \\
* & M_{(a_2-a_1) \times (a_2-a_1)} & * & \\
* & * & \ddots & * \\
* & * & * & M_{(n-a_k) \times (n-a_k)} \end{pmatrix}
\]

The Levi component is

\[
m_S = \{(x_0, \ldots, x_k) \in \mathfrak{gl}_{a_1} \times \cdots \times \mathfrak{gl}_{a-n-a_k} : \text{tr}(x_0) + \cdots + \text{tr}(x_k) = 0\}
\]

\[
\simeq \mathfrak{sl}_{a_1} \times \cdots \times \mathfrak{sl}_{a-n-a_k} \times \mathbb{C}^\oplus k.
\]

First, we must extend relevant definitions from simple Lie algebras to reductive Lie algebras. The following is the generalization of the critical level:

**Definition 1.4.** Let \( g \) be a reductive Lie algebra, which decomposes as \( g = \bigoplus_{i=1}^{s} \mathfrak{g}_i \oplus \mathfrak{g}_0 \) for some simple Lie algebras \( \mathfrak{g}_1, \ldots, \mathfrak{g}_s \) and an abelian Lie algebra \( \mathfrak{g}_0 \). Then the critical level is \( \kappa_c(g) := (\kappa_{i,c})_{i=0}^{s} \), where \( \kappa_{0,c} = 0 \) and \( \kappa_{i,c} \) is the critical level for the simple Lie algebra \( \mathfrak{g}_i \) for \( 1 \leq i \leq s \).

Given an invariant symmetric bilinear form \( \kappa \) on \( g \), let \( \widehat{g}_k \) be the corresponding affine Kac-Moody algebra, as in [KL24]: it is the central extension

\[
0 \to \mathbb{C} \to \widehat{g}_k \to g((t)) \to 0
\]

with commutation relation

\[
[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\kappa(A, B) \text{Res} f dg) 1.
\]

Let us now formally re-state our goal:

**Goal 1.5.** Let \( g \) be a reductive Lie algebra, let \( \kappa \) be an invariant symmetric bilinear form on \( g \), and let \( p = m \times u \subset g \) be a parabolic subalgebra. Define an exact functor

\[
\widetilde{U}_{\kappa|m+\kappa(\widehat{m})}(m)\text{-mod} \to U_{\kappa+\kappa_c}(\widehat{g})\text{-mod}
\]

such that the Wakimoto module with highest weight \( \lambda \) is sent to the Wakimoto module with highest weight \( \lambda \).

1.1. **Finite-dimensional analog.** Let us first describe the finite-dimensional analog of Goal 1.5.

**Definition 1.6.** Let \( g \) be a simple Lie algebra with standard parabolic subalgebra \( p = m \times u \). There is an exact functor, the parabolic induction functor

\[
\text{Ind}_p^g : m\text{-mod} \to g\text{-mod}.
\]

Given a \( m \)-module \( V \), we may view it as a \( p \)-module by extension by zero, i.e., by making \( u \) act by zero, and we let

\[
\text{Ind}_p^g V := U(g) \otimes_{U(p)} V.
\]

Now the \( \text{Ind}_p^g \) sends Verma modules to Verma modules:

**Lemma 1.7.** For a weight \( \lambda \in \mathfrak{h}^* \), let \( V_m(\lambda) \) and \( V_g(\lambda) \) be the Verma modules with highest weight \( \lambda \) of the Lie algebras \( m \) and \( g \), respectively. Then

\[
\text{Ind}_p^g V_m(\lambda) \simeq V_g(\lambda).
\]
\textbf{Proof.} Follows from observing that $U(p) \otimes U(b_+) \mathbb{C}_\lambda$ is isomorphic to the inflation of the $m$-module $V_m(\lambda)$ to $p$, and because induction is transitive. \hfill \Box

\textbf{Remark 1.8.} When $p = b_+$, the above recovers the construction of Verma modules (i.e., $V_p(\lambda) = \text{Ind}^g_{b_+} \mathbb{C}_\lambda$).

Recall that [Kiy24] gives a geometric construction of dual Verma modules using vector fields on $G/N_-$, where $N_-$ is the unipotent radical of the opposite Borel subalgebra $B_-$. The construction admits a straightforward generalization to the parabolic setting: let $P_{\pm} = M \ltimes U_{\pm} \subset G$ be subgroups whose Lie algebras are $p_{\pm} = m \ltimes u_{\pm} \subset g$. Then analogously to [Kiy24, §2] there is a map of Lie algebras
\[ g \to \text{Vect}(G/U_-)^{M_r}, \]
where $M_r$ acts on $G/U_-$ from the right.\footnote{Here, the action is well-defined because $M$ normalizes $U_-$.} Now as in Daishi’s talk, $P_+ U_- / U_- \subset G/U_-$ is Zariski open, and restricting to the locus gives a homomorphism of algebras
\[ (1.9) \quad \varphi_{U_+} : U(g) \to D(P_+)^M \simeq D(U_+) \otimes U(m), \]
where the second isomorphism follows from the isomorphism of varieties $P_+ \simeq U_+ \times M$. Now:

\textbf{Lemma 1.10.} Let $V$ be a $m$-module, with structure morphism $\varphi : U(m) \to \text{End}(V)$. Then the modified $g$-module structure on $\mathbb{C}[U_+] \otimes V$ is defined by
\[ U(g) \to D(U_+) \otimes U(m) \xrightarrow{1 \otimes \varphi} D(U_+) \otimes \text{End}(V) \to \text{End}(\mathbb{C}[U_+] \otimes V), \]
noting that $\mathbb{C}[U_+]$ is naturally a $D(U_+)$-module. Then the $g$-module $\mathbb{C}[U_+] \otimes V^\vee$ is isomorphic to the dual parabolic induction $\text{Ind}^{\mathbb{C}[U_+]}_m V^\vee$.\footnote{Here, as usual, letting $\tau : g \to g$ be the Cartan involution and given $M = \bigoplus \mu M_\mu$, we let $M^\vee = \bigoplus \mu M_\mu^\vee$ with $\langle x, n, m \rangle = \langle n, -\tau(x)m \rangle$ for $n \in M^\vee, m \in M$. Alternatively, it is the \textit{parabolic co-induction}, the right adjoint to restriction.}

We hope to see Lemma 1.7 from the geometric perspective:

\textbf{Proposition 1.11.} Let $P_+ = M \ltimes U_+ \subset G$ be a standard parabolic subgroup. There is a commutative diagram
\[
\begin{array}{ccc}
U(g) & \xrightarrow{\varphi_{U_+}} & D(N_+) \otimes U(h) \\
\downarrow{\varphi_{U_+}} & & \downarrow{\simeq} \\
D(U_+) \otimes U(m) & \xrightarrow{id_D(U_+) \otimes \varphi_{U_+} \cap M} & D(U_+) \otimes (D(N_+ \cap M) \otimes U(h)).
\end{array}
\]
Here, the homomorphisms $U(g) \to D(N_+) \otimes U(h)$ and $U(g) \to D(U_+) \otimes U(m)$ are as in (1.9), and the right vertical isomorphism follows from the multiplication isomorphism\footnote{An isomorphism of varieties; not of groups!} $U_+ \times (N_+ \cap M) \simeq N_+$.

\textbf{Proof.} Indeed, the following diagram commutes:
\[
\begin{array}{ccc}
D(G)^{G_r} & \xrightarrow{\longrightarrow} & D(G/U_-)^{M_r} & \xrightarrow{\longrightarrow} & D(G/N_-)^{H_r} \\
\downarrow & & \downarrow & & \downarrow \\
D(P_+)^{M_r} & \xrightarrow{\longrightarrow} & D(P_+/(P_+ \cap N_-))^ {H_r}
\end{array}
\]
where the vertical homomorphisms are restricting along open immersions $P_+ \subset G/U_-$ and $P_+/(P_+ \cap N_-) \subset G/N_-$. The first horizontal homomorphism $D(G)^{G_r} \hookrightarrow D(G/U_-)^{M_r}$ is obtained as follows: any $\sigma \in D(G)^{G_r}$ is an operator $\sigma : \mathbb{C}[G] \to \mathbb{C}[G]$ which is $G_r$-invariant, hence it sends $(U_-)_r$-invariant functions to $(U_-)_r$-invariant functions. In fact, for any $(U_-)_r$-invariant open subset $X$ of...
G, there is an operator \( \sigma: \mathbb{C}[X]^{U-r} \rightarrow \mathbb{C}[X]^{U-r} \). In other words, since \( \mathbb{C}[X/U_-] = \mathbb{C}[X]^{U-r} \), it defines an endomorphism of sheaves \( \tilde{\sigma}: \mathcal{O}_{G/U_-} \rightarrow \mathcal{O}_{G/U_-} \), which can be shown to be a differential operator. Note that we need \( \tilde{\sigma} \) to be an endomorphism of the sheaf \( \mathcal{O}_{G/U_-} \), and not just \( \mathbb{C}[G/U_-] \), since \( G/U_- \) may not be affine, e.g., \( \text{SL}_2/N_- \simeq \mathbb{A}^2 \setminus \{(0,0)\} \). Moreover \( \sigma \) is \( G_r \)-invariant so \( \tilde{\sigma} \) must be \( M_r \)-invariant, i.e., \( \tilde{\sigma} \in D(G/U_-)^{M_r} \). All other horizontal maps are constructed in a similar fashion.

Now we have the isomorphisms \( U(\mathfrak{g}) \simeq D(G)^{G_r} \) and \( D(P_+)^{M_r} \simeq D(U_+) \otimes U(\mathfrak{h}) \), so (1.12) can be re-written as

\[
\begin{array}{cccc}
U(\mathfrak{g}) & \xrightarrow{\varphi_{B^r}} & D(G/U_-)^{M_r} & \xrightarrow{\varphi^G_{P_+}} & D(G/N_-)_r' \xrightarrow{D(N_+) \otimes U(\mathfrak{h})} \end{array}
\]

which is the desired commutativity. Here the homomorphism \( D(G/N_-)_r' \rightarrow D(N_+) \otimes U(\mathfrak{h}) \) is the composition of the restriction to the open Bruhat cell \( D(G/N_-)_r' \rightarrow D(B_+)^{M_r} \), together with the standard isomorphism \( D(B_+)^{H_r} \simeq D(N_+) \otimes U(\mathfrak{h}) \) from [Kiy24].

**Remark 1.13.** Proposition 1.11 implies Lemma 1.7.

### 1.2. Back to affine Lie algebras.

Recall the definition of the Weyl algebra \( \tilde{\Gamma}^\theta \) (denoted simply as \( \tilde{\Gamma} \) in [Wan24b]):

**Definition 1.14.** Let \( \tilde{\Gamma}^\theta = \mathbb{C}1 + \mathfrak{n}_+((t)) \oplus \mathfrak{n}_-^*((t))dt \) with Lie bracket

\[
[f, g] = \langle x, y \rangle \text{Res}(f) \cdot 1
\]

for \( x \in \mathfrak{n}_+, y \in \mathfrak{n}_-^* \), and \( f \in \mathbb{C}((t)) \). More concretely, it has a topological basis \( 1 \), \( a_{\alpha,n} := x_{\alpha} t^n \), and \( a_{\alpha,n}^* := x_{\alpha}^* t^n dt \) for \( \alpha \in \Delta_+ \) and \( n \in \mathbb{Z} \) with relations

\[
[a_{\alpha,n}, a_{\beta,m}^*] = \delta_{\alpha, \beta} \delta_{m+n,0} 1 \quad \text{and} \quad [a_{\alpha,n}, a_{\beta,m}] = [a_{\alpha,n}, a_{\beta,m}^*] = 0.
\]

Let \( \tilde{\Gamma}^\theta_+ = \mathfrak{n}_+[[t]] \oplus \mathfrak{n}_-^*[[t]]dt \), i.e., the abelian subalgebra with topological basis \( a_{\alpha,n} \) for \( n \geq 0 \) and \( a_{\alpha,n}^* \) for \( n > 0 \).

Given a invariant symmetric bilinear form \( \kappa \) on \( \mathfrak{g} \), define the affine vertex algebra \( V_\kappa(\mathfrak{g}) = \text{Ind}_{\mathfrak{g}[t]}^{\tilde{\Gamma}^\theta} \mathbb{C}_{\kappa} \) by the same formulas as in [Dum24]: for \( x \in \mathfrak{g} \),

\[
\mathcal{Y}(xt^{-1}|0), z) = \sum_{n \in \mathbb{Z}} xt^n z^{-n-1}
\]

and \([T, xt^n] = -nx t^{n-1}\). When \( \mathfrak{g} \) decomposes as a direct sum, the affine vertex algebra decomposes as a tensor product:

**Lemma 1.17.** Let \( \mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i \oplus \mathfrak{g}_0 \) where \( \mathfrak{g}_1, \ldots, \mathfrak{g}_s \) are simple Lie algebras and \( \mathfrak{g}_0 \) is abelian. Then there is an isomorphism

\[
V_\kappa(\mathfrak{g}) \simeq \bigotimes_{i=0}^s V_\kappa(\mathfrak{g}_i),
\]

where:

\[\text{[Fre07, §5.3.3]} \text{ denotes this Lie algebra as } \mathfrak{A}^\theta, \text{ but we avoid this notation since in [Kiy24] it denotes an associative algebra. In our notes, } \mathfrak{A}^\theta \text{ denotes an associative algebra with the same relations as } \tilde{\Gamma}^\theta.\]
• $V_{\kappa_1}(\frak{g}) = \text{Ind}_{\frak{g}_0[1]}^{\frak{g}_0[1]+1} \mathbb{C}[0]$ is the vacuum module over $\frak{g}_{\kappa_1}$ with the vertex algebra structure given as in [Dun24].

• $V_{\kappa_0}(\frak{g}_0) = \text{Ind}_{\frak{g}_0[0]}^{\frak{g}_0[1]} \mathbb{C}[0]$ is the Fock representation of the Heisenberg algebra $\frak{g}_0$.

Remark 1.26. When small make this precise below.

Definition 1.27. Let $\frak{g}$ be a parabolic subalgebra. Then there exists a map of vertex algebras

\[ w_\kappa : V_{\kappa+\kappa_1}(\frak{g}) \to M_\frak{g} \otimes V_\kappa(\frak{h}), \]

where $M_\frak{g} = \text{Ind}_{\frak{g}_0[0]}^{\frak{g}_0[1]} \mathbb{C}[0]$ is the Fock representation of the Weyl algebra $\frak{g}$ and a vertex algebra, i.e., it is generated by a vector $|0\rangle$ such that

\[ a_{\alpha,n}|0\rangle = 0 \text{ for } n \geq 0, a_{\alpha,n}^*|0\rangle = 0 \text{ for } n > 0, \text{ and } |1\rangle|0\rangle = |0\rangle. \]

Later, we will use the explicit formula for $w_{\kappa_c}$, as stated in [Wan24b, §4] and [Fre07, Theorem 6.2.1]:

Theorem 1.20. The homomorphism of vertex algebras $w_{\kappa_c} : V_{\kappa_c}(\frak{g}) \to M_\frak{g} \otimes V_0(\frak{h})$ is explicitly,

\[ w_{\kappa_c}(e_1(z)) = a_{\alpha_1}(z) + \sum_{\beta \in \Delta_+} P_\beta^c(\frak{g}^c(\frak{g})) a_{\beta}(z); \]

\[ w_{\kappa_c}(h_1(z)) = - \sum_{\beta \in \Delta_+} \beta(h_1) a_{\beta}^c(z) a_{\beta}(z) + b_1(z) \]

\[ w_{\kappa_c}(f_1(z)) = \sum_{\beta \in \Delta_+} Q_\beta^c(\frak{g}^c(\frak{g})) a_{\beta}(z) + b_1(z) a_{\alpha_1}^c(z) + c_i \partial_i a_{\alpha_i}(z), \]

for some constants $c_i \in \mathbb{C}$, where $P_\beta^c$ and $Q_\beta^c$ are explicit polynomials defined in [Fre07, §5.2].

By the isomorphism $\tilde{U}(V_{\kappa}(\frak{g})) \simeq \tilde{U}_{\kappa}(\frak{g})$ from [Wan24a, §2.3], the homomorphism $w_\kappa$ induces a map on the completed universal enveloping algebras

\[ \tilde{U}_{\kappa+\kappa_1}(\frak{g}) \to \mathfrak{g}^c \otimes \tilde{U}_{\kappa_1}(\frak{h}). \]

We hope to generalize the homomorphism $w_{\kappa}$ to arbitrary parabolics. Our goal is to prove the following, which is the affine analog of the homomorphism (1.19):

Theorem 1.25. Let $\kappa$ be an invariant symmetric bilinear form on $\frak{g}$, and let $\frak{p} \subset \frak{g}$ be a parabolic subalgebra. Then there exists a map of vertex algebras

\[ w_\kappa^\frak{p} : V_{\kappa+\kappa_1}(\frak{g}) \to M_{\frak{g}_0^\frak{p}} \otimes V_{\kappa_1}(\frak{m}) \]

Here, $M_{\frak{g}_0^\frak{p}}$ is also a Weyl vertex algebra, but for a smaller nilpotent Lie algebra than $\frak{n}_+$. We small make this precise below.

Remark 1.26. When $\frak{p} = \frak{b}_+$, we have $w_\kappa^\frak{p} = w_\kappa$ from (1.18).

Let us define first all the notation in the theorem statement.

Let $\Delta_+^\prime$ be the set of positive roots of $\frak{g}$ occurring in $\frak{u}_+$, or, equivalently, not occuring in $\frak{p}_-$. The following is the generalization of $\tilde{\frak{g}}^c$ to the parabolic setting:

Definition 1.27. Let $\tilde{\frak{g}}^c = \mathbb{C} \oplus \frak{u}_+(t) \oplus \frak{u}_-^c(t) dt$ with Lie bracket as in (1.15). Explicitly, it has topological basis $1, a_{\alpha,n}, a_{\alpha,n}^*$ for $\alpha \in \Delta_+^\prime$ and $n \in \mathbb{Z}$, with brackets

\[ [a_{\alpha,n}, a_{\beta,m}^*] = \delta_{\alpha,\beta} \delta_{m+n,0} 1 \]

\[ [a_{\alpha,n}, a_{\beta,m}] = [a_{\alpha,n}^*, a_{\beta,m}^*] = 0. \]

There is a sub-Lie algebra $\tilde{\frak{g}}_+^c := \frak{u}_+[t] \oplus \frak{u}_-^c[t] dt$. Let the Fock representation be $M_{\frak{g}_0^\frak{p}} = \text{Ind}_{\tilde{\frak{g}}_+^c}^{\frak{g}_0^\frak{p}}(\mathbb{C}[0]).$
The Fock representation \( M_{g,p} \) can be given a vertex algebra structure by the same formula used for \( M_g \). It is related to \( M_g \) as follows:

**Exercise 1.28.** There is a vertex algebra isomorphism

\[
M_{g,p} \otimes M_m \simeq M_g,
\]

sending:

\[
a_{\alpha,n}(0) \otimes (0) \mapsto a_{\alpha,n}(0), \quad a_{\alpha,n}^\ast(0) \otimes (0) \mapsto a_{\alpha,n}^\ast(0) \quad \text{for} \quad \alpha \in \Delta_+^\prime, \quad \text{and}
\]

\[
(0) \otimes a_{\beta,n}(0) \mapsto a_{\beta,n}(0), \quad (0) \otimes a_{\beta,n}^\ast(0) \mapsto a_{\beta,n}^\ast(0) \quad \text{for} \quad \alpha \in \Delta_+ \setminus \Delta_+^\prime.
\]

The proof of Theorem 1.25 follows the same strategy as [Fre07, Theorem 6.2.1], explained by [Wan24b], so we will not repeat it here.

Now Theorem 1.25 gives a homomorphism analogous to (1.24):

\[
\tilde{U}_{\kappa+\kappa_c}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{O}_{\mathfrak{g}}^{-\mathfrak{p}} \otimes U_{\kappa|m+\kappa_c(m)}(\bar{m}),
\]

which allows us to define generalized Wakimoto modules:

**Definition 1.29.** Let \( R \) be a smooth \( \tilde{m}_{\kappa|m+\kappa_c(m)} \)-module. Then \( M_{g,p} \otimes R \) carries a smooth \( \widehat{\mathfrak{g}}_{\kappa+\kappa_c} \)-module structure, called the **generalized Wakimoto module** corresponding to \( R \). We denote it by \( \text{Wak}_p^\mathfrak{g}R \).

Now we have the following analog of Lemma 1.7, which finally accomplishes Goal 1.5 (see [Los24b] for a proof sketch):

**Proposition 1.30.** There is a commutative diagram:

\[
\begin{array}{ccc}
V_{\kappa+\kappa_c}(\mathfrak{g}) & \xrightarrow{u_{\kappa}} & M_{\mathfrak{g}} \otimes V_{\kappa|0}(\mathfrak{h}) \\
\downarrow u^\mathfrak{g}_\kappa & & \downarrow \simeq \\
M_{g,p} \otimes V_{\kappa|m+\kappa_c(m)}(\mathfrak{m}) & \xrightarrow{1 \otimes u_{\kappa|0}} & M_{g,p} \otimes M_m \otimes V_{\kappa|0}(\mathfrak{h}),
\end{array}
\]

where the vertical isomorphism was defined in Exercise 1.28. Thus, for any \( \lambda \in \mathfrak{h}^* \) there is an isomorphism

\[
\text{Wak}_p^\mathfrak{g}(W_{\lambda,\kappa|m+\kappa_c(m)}) \simeq W_{\lambda,\kappa+\kappa_c}.
\]

2. **Comparing affine Verma modules to Wakimoto modules**

Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra now. Let \( \tilde{\mathfrak{b}}_+ := \mathfrak{b}_+ + t\mathfrak{g}[t] \) and \( \tilde{\mathfrak{n}}_+ := \mathfrak{n}_+ + t\mathfrak{g}[t] \) be the pre-images of \( \mathfrak{b}_+ \) and \( \mathfrak{n}_+ \), respectively, under the quotient map \( \mathfrak{g}[t] \rightarrow \mathfrak{g} \) evaluating at \( t = 0 \). The subalgebra \( \tilde{\mathfrak{b}}_+ \) is called the **Iwahori subalgebra**, and \( \tilde{\mathfrak{n}}_+ \) is its topological nilpotent radical. Now for a weight \( \lambda \in \mathfrak{h}^* \) let \( \mathbb{C}_\Lambda \) be the one-dimensional representation of \( \tilde{\mathfrak{b}}_+ \oplus \mathbb{C}1 \) such that \( \tilde{\mathfrak{n}}_+ \) acts by zero, \( \mathfrak{h} \) acts by \( \lambda \), and \( 1 \) acts as the identity.

**Definition 2.1.** The **Verma module** \( M_{\mathfrak{b},\kappa} \) of level \( \kappa \) and highest weight \( \lambda \) is

\[
M_{\mathfrak{b},\kappa} := \text{Ind}_{\mathfrak{b}_+ \oplus \mathbb{C}1}^{\tilde{\mathfrak{b}}_+} \mathbb{C}_\Lambda.
\]

Denote the highest-weight vector, \( 1 \otimes 1 \), as \( v_{\lambda,\kappa} \).

We hope to compare the Wakimoto module \( W_{0,\kappa_c} \) with the Verma module \( M_{0,\kappa_c} \). There is a homomorphism

\[
M_{0,\kappa_c} \rightarrow V_{\kappa_c}(\mathfrak{g}) \xrightarrow{w_{\kappa_c}} W_{0,\kappa_c}.
\]
which sends the highest-weight vector $v_{0,\kappa}$ to $|0\rangle \otimes |0\rangle$, since by construction $w_{\kappa}$ is $\mathfrak{g}_{\kappa}$-equivariant. Here, the first homomorphism is by the transitivity of induction:

$$M_{0,\kappa} = \text{Ind}_{\mathfrak{g}}^{\mathfrak{b}} \mathcal{C}_0 = \text{Ind}_{\mathfrak{g}[t] \oplus \mathfrak{c}_1}^{\mathfrak{b}[t] \oplus \mathfrak{c}_1} \text{Ind}_{\mathfrak{b}}^\mathfrak{g} \mathcal{C}_0 \to \text{Ind}_{\mathfrak{g}[t] \oplus \mathfrak{c}_1}^{\mathfrak{b}[t] \oplus \mathfrak{c}_1} \mathcal{C} = V_{\kappa}(\mathfrak{g}).$$

However, (2.2) cannot be an isomorphism; indeed, the energy zero component of $M_{\lambda,\kappa}$ is the Verma module $\text{Ind}_{\mathfrak{b}}^\mathfrak{g} \mathcal{C}_0$ while the energy zero component of $W_{0,\kappa}$ is the dual Verma module $(\text{Ind}_{\mathfrak{b}}^\mathfrak{g} \mathcal{C}_0)^\vee$, so they cannot be isomorphic. Thus, we modify the Wakimoto modules $W_{\lambda,\kappa}$ to $W_{\lambda,\kappa}^+$ to be defined below, so that the following holds:

**Theorem 2.3** ([Fre07, Proposition 6.3.3]). The Wakimoto module $W_{0,\kappa}^+$ is isomorphic to the Verma module $M_{0,\kappa}$.

To define $W_{\lambda,\kappa}^+$, the Fock representation of $\tilde{\Gamma}^0$, defined as $M_0 := \text{Ind}_{\tilde{\Gamma}^0}^{\mathfrak{b}[t] \oplus \mathfrak{c}_1} \mathcal{C}|0\rangle$, is modified to the module with the following modification of (1.19):

$$a_{\alpha,n}|0\rangle' = 0 \text{ for } n > 0, \quad a^*_{\alpha,n}|0\rangle' = 0 \text{ for } n \geq 0, \quad \text{and } 1|0\rangle' = |0\rangle'.$$

Now let

$$(4.4) \quad W_{\lambda,\kappa}^+ := M'_0 \otimes \pi_{-2\rho-\lambda},$$

where $\pi_{-2\rho-\lambda}$ was defined in [Wan24b, §0]:

$$\pi_{-2\rho-\lambda} := \text{Ind}_{\mathfrak{b}[t] \oplus \mathfrak{c}_1}^{\mathfrak{b}[t] \oplus \mathfrak{c}_1} \mathcal{C}| -2\rho-\lambda).$$

Denote the vector $|0\rangle' \otimes | -2\rho-\lambda)$ in $W_{\lambda,\kappa}^+$ as $|0\rangle'$. The shift by $2\rho$ in (4.4) is explained in §2.2; it is necessary for $|0\rangle' \in W_{\lambda,\kappa}^+$ to be a highest weight vector of weight $\lambda$.

We may modify the formulas in Theorem 1.20 to obtain a homomorphism $\tilde{\mathfrak{g}}_{\kappa}$-module structure

on $W_{\lambda,\kappa}^+$. We will give explicit formulas at the critical level:

**Theorem 2.5.** The module $W_{\lambda,\kappa}^+$ has a $\tilde{\mathfrak{g}}_{\kappa}$-module structure given by

$$(6.6) \quad w_{\kappa}(f_{\alpha}(z)) = a_{\alpha}(z) + \sum_{\beta \in \Delta_+} \beta^i(\alpha^i(z))a_{\beta}(z):$$

$$(6.7) \quad w_{\kappa}(h_i(z)) = \sum_{\beta \in \Delta_+} \beta(h_i):a^*_{\beta}(z)a_{\beta}(z): -b_i(z)$$

$$(6.8) \quad w_{\kappa}(e_i(z)) = \sum_{\beta \in \Delta_+} Q^i_{\beta}(\alpha^i(z))a_{\beta}(z): +b_i(z)a^*_{\alpha}(z) + c_i\partial_za^*_{\alpha}(z),$$

for some constants $c_i \in \mathbb{C}$ and where polynomials $P^i_{\beta}$ and $Q^i_{\beta}$ are defined in [Fre07, §5.2].

In fact, there are formulas for $w_{\kappa}(f_{\alpha}(z))$ for arbitrary $\alpha \in \Delta_+$, not just for simple roots:

$$(6.9) \quad w_{\kappa}(f_{\alpha}(z)) = a_{\alpha}(z) + \sum_{\beta \in \Delta_+; \beta > \alpha} \beta^i(\alpha^i(z))a_{\beta}(z):$$

for some polynomials $P^i_{\beta}$. See [Fre07, equation (6.1-2)].

We prove Theorem 2.3 in three steps:

(a) comparing the formal characters;

(b) constructing a homomorphism $M_{0,\kappa} \to W_{0,\kappa}^+$; and

(c) proving the surjectivity of the homomorphism.

From the three steps, the isomorphism is clear: the character of the kernel of $M_{0,\kappa} \to W_{0,\kappa}^+$ must be zero by (a). Step (b) is accomplished in exactly the same way the homomorphism $M_{0,\kappa} \to W_{0,\kappa}$ was constructed in (2.2), by sending the highest vector $v_{0,\kappa}$ to the vacuum vector $|0\rangle' \otimes | -2\rho)$. 

2.1. Formal characters of $\bar{U}_\kappa(\mathfrak{g})$-modules. To check (a), let us recall what the character of a $\bar{U}_\kappa(\mathfrak{g})$-module is. For a $\bar{U}_\kappa(\mathfrak{g})$-module $M$, suppose there is a grading operator $d: M \to M$ compatible with the $\mathfrak{g}_\kappa$-action, i.e., such that $[d, xt^n] = n xt^{n-1}$. Let $\mathfrak{h}' = \mathfrak{h} \oplus \mathbb{C}d$, so the characters are of the form $\chi' = (\lambda, \phi)$ where $\lambda \in \mathfrak{h}'^*$ and $\phi \in \mathbb{C}$, so $d$ acts by $\phi$.

Now, we can define the character of a $\bar{U}_\kappa(\mathfrak{g})$-module:

**Definition 2.10.** Let $M$ be a smooth $\bar{U}_\kappa(\mathfrak{g})$-module, such that 1 acts by identity and the Cartan $\mathfrak{h} \oplus \mathbb{C}d \oplus \mathbb{C}1$ acts semi-simply on $M$ with finite-dimensional weight spaces:

$$M = \bigoplus_{\lambda' \in (\mathfrak{h}')^*} M(\lambda').$$

Then the character of $M$ is

$$\text{ch} M = \sum_{\lambda' \in (\mathfrak{h}')^*} \dim M(\lambda') \cdot e^{\lambda'}.$$

Let $\delta := (0,1) \in \mathfrak{h}_1^*$, the set of positive roots of $\mathfrak{g}'$ is:

$$\hat{\Delta}_+ = (\Delta_+ + \mathbb{Z}_{\geq 0}\delta) \cup (\{0\} + \mathbb{Z}_{\geq 0}\delta).$$

The positive roots define a partial order on $\mathfrak{h}_1^*$:

**Definition 2.12.** Let $\lambda' > \mu'$ if $\lambda' - \mu' = \sum_i \beta'_i$ for some $\beta'_i \in \hat{\Delta}_+$.

The Verma module $\mathbb{M}_{\lambda,\kappa}$ over $\mathfrak{g}_\kappa$, as defined in Definition 2.1, can be extended to $\mathfrak{g}_\kappa'$, which we denote by $\mathbb{M}_{\lambda',\kappa}$ where $\lambda' = (\lambda,0)$:

$$\mathbb{M}_{\lambda',\kappa} := \text{Ind}_{\mathfrak{h}'(\mathfrak{g}^\kappa) \oplus \mathbb{C}1 \oplus \mathbb{C}d} \mathbb{C}^{\lambda'}.$$

Now by the PBW theorem, as a vector space $\mathbb{M}_{\lambda,\kappa} \simeq U(\bar{n}_-)$, where $\bar{n}_- = n_- \oplus t^{-1} \mathfrak{g}[t^{-1}]$, so

$$\text{ch} \mathbb{M}_{\lambda',\kappa} = e^\lambda' \prod_{\alpha' \in \Delta_+} (1 - e^{-\alpha'})^{-\text{mult} \alpha'},$$

where mult $\alpha'$ is the dimension of the weight space $\mathfrak{g}_{\kappa_0,\alpha'}$.

Since $W_{0,\kappa_0}^+$ has a basis in the monomials

$$a_{\alpha,n}, \alpha \in \Delta_+, n < 0; a^*_{\alpha,n}, \alpha \in \Delta_+, n \leq 0; \text{ and } b_{i,n}, i = 1, \ldots, \ell, n < 0,$$

to compute the character of the $\mathfrak{g}_\kappa$-module $W_{0,\kappa_0}^+$, we must compute the $\mathfrak{h}'$-action on $a_{\alpha,n}, a^*_{\alpha,n}$, and $b_{i,n}$.

Since $d$ simply acts by $L_0 = -t \partial_0$ on $M_0 \otimes V_0(\mathfrak{h})$,

$$[d, a_{\alpha,n}] = -n a_{\alpha,n}, \quad [d, a^*_{\alpha,n}] = -n a^*_{\alpha,n}, \quad [d, b_{i,n}] = -nb_{i,n},$$

where $a_{\alpha,n}, a^*_{\alpha,n} \in M_0$, and $b_{i,n} \in \bar{U}_0(\mathfrak{h}(t))$. The $\mathfrak{h}$-action on $W_{0,\kappa_0}^+$ is given by, for $h \in \mathfrak{h}$,

$$[h, a_{\alpha,n}] = \alpha(h) a_{\alpha,n}, \quad [h, a^*_{\alpha,n}] = \alpha(h) a^*_{\alpha,n}, \quad [h, b_{i,n}] = 0.$$}

Formula (2.16) follows from (1.22):

**Exercise 2.17.** Deduce formula (2.16) from (1.22).

Now (2.15) and (2.16) together show that the character of $W_{0,\kappa_0}^+$ equals (2.13).
2.2. Constructing the homomorphism. Let us compute the action of $\mathfrak{h}$ on $|0\rangle'$:

**Exercise 2.18.** For any $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$, then $h \cdot |0\rangle' = \lambda(h)|0\rangle'$ in $W_{0,\kappa_c}^+$.

The Exercise shows why the shift by 2$\rho$ was necessary in (2.4). The classical analog is the following: $\mathbb{C}[x]$ and $\mathbb{C}[\delta_0]$ are both $D(\mathbb{A}^1) \simeq \mathbb{C}[x, \partial_x]$-modules, where $\delta_0$ is the delta function supported on 0. Then $L_0 = -x\partial_x$ acts as 0 on 1 $\in \mathbb{C}[x]$, but acts instead as $-x\partial_x \cdot 1 = (1 - \partial_x)1 = 1$.

**Solution to Exercise 2.18.** The constant term in (2.7) is (by definition of the normally ordered product)

$$w_{\kappa_c}(h_1,0|0\rangle) = \sum_{\beta \in \Delta_+} \beta(h_i) \left( \sum_{n \geq 0} a_{\beta,-n}^* a_{\beta,n} + \sum_{n < 0} a_{\beta,n} a_{\beta,-n}^* \right) |0\rangle' - b_{i,0} |0\rangle'$$

$$= \sum_{\beta \in \Delta_+} \beta(h_i) a_{\beta,0}^* a_{\beta,0} |0\rangle' - (-2\rho - \lambda)(h_i)|0\rangle'$$

$$= \sum_{\beta \in \Delta_+} \beta(h_i) (a_{\beta,0}^* a_{\beta,0} - 1)|0\rangle' + (2\rho + \lambda)(h_i)|0\rangle'$$

$$= - \sum_{\beta \in \Delta_+} \beta(h_i)|0\rangle' + (2\rho + \lambda)(h_i)|0\rangle'$$

$$= \lambda(h_i)|0\rangle'. \quad \square$$

Now, by the character formula in §2.1 the weight spaces of $\lambda' > 0$ are zero, i.e., $|0\rangle' \in W_{0,\kappa_c}^+$ is annihilated by $\tilde{n}_+$. Thus there is a homomorphism $M_{0,\kappa_c} \to W_{0,\kappa_c}^+$.

2.3. Proving the surjectivity of the homomorphism.

**The remainder of the proof of Theorem 2.3.** We need to check that $M_{0,\kappa_c} \to W_{0,\kappa_c}^+$ is surjective, i.e., that $W_{0,\kappa_c}^+$ is generated as a $\tilde{\mathfrak{g}}_{\kappa_c}$-module by $|0\rangle'$. Consider the coinvariants of $W_{0,\kappa_c}^+$ with respect to $\tilde{n}_- = n_- \oplus t^{-1} \mathfrak{g}[t^{-1}]$:

$$(W_{0,\kappa_c}^+)_{\tilde{n}_-} := C_0 \otimes_{U(n_-)} W_{0,\kappa_c}^+,$$

which is a $\mathfrak{h}'$-representation since $\tilde{n}_- \subset \tilde{\mathfrak{g}}_{\kappa_c}$ is $\mathfrak{h}'$-stable. If $M_{0,\kappa_c} \to W_{0,\kappa_c}^+$ were not surjective, then there is an exact sequence of $\tilde{\mathfrak{g}}_{\kappa_c}$-modules

$$M_{0,\kappa_c} \to W_{0,\kappa_c}^+ \to V \to 0,$$

for some non-zero $V$, which induces an exact sequence of $\mathfrak{h}'$-modules

$$M_{0,\kappa_c} \to W_{0,\kappa_c}^+ \to V \to 0,$$

(2.19) where $V_{\tilde{n}_-} \neq 0$. But $(M_{0,\kappa_c})_{\tilde{n}_-} \to (W_{0,\kappa_c}^+)_{\tilde{n}_-}$ is an isomorphism on the $(0,0)$-weight space, so $(W_{0,\kappa_c}^+)_{\tilde{n}_-}$ must have a nonzero weight $\mu'$. In other words $W_{0,\kappa_c}^+$ has an irreducible quotient $L_{\mu',\kappa_c}$ with highest weight $\mu'$. Since $M_{0,\kappa_c}$ and $W_{0,\kappa_c}^+$ have the same characters, they define the same class in the Grothendieck group and hence must have the same irreducible subquotients. Now we shall:

1. observe restrictions on $\mu' \in (\mathfrak{h}')^*$ coming from $L_{\mu',\kappa_c}$ being a subquotient of $W_{0,\kappa_c}^+$; and
2. observe restrictions on $\mu' \in (\mathfrak{h}')^*$ coming from $L_{\mu',\kappa_c}$ being a subquotient of $M_{0,\kappa_c}$.

---

8They are Fourier transforms of each other.
We will show the two restrictions on $\mu'$ are incompatible, and hence our assumption, that $V \neq 0$, must have been wrong.

First, however, there is a subtlety: $M_{0,\kappa}$ and $W^+_{0,\kappa}$ have infinite length, so $\text{ch} M_{0,\kappa} = \text{ch} W^+_{0,\kappa}$ does not imply they have the same irreducible subquotients in the naïve way. The correct statement is as follows:

Exercise 2.20. Let $M$ and $N$ be category $\mathcal{O}$-modules for $\mathfrak{g}'_\kappa$. Then $\text{ch} M = \text{ch} N$ if and only if $M$ and $N$ define the same class in the completed Grothendieck group $\hat{K}_0(\mathcal{O}_{\mathfrak{g}'_\kappa})$, which is the inverse limit

$$
\hat{K}_0(\mathcal{O}_{\mathfrak{g}'_\kappa}) := \lim_{\lambda \in (\mathfrak{h}')^*} K_0(\mathcal{O}_{\mathfrak{g}'_\kappa}/\mathcal{O}_{\mathfrak{g}'_\kappa, \leq \lambda}),
$$

over the partial order on $(\mathfrak{h}')^*$ defined in (2.12) where $\mathcal{O}_{\mathfrak{g}'_\kappa, \leq \lambda}$ is the Serre subcategory of $\mathcal{O}_{\mathfrak{g}'_\kappa}$ consisting of modules with weights $\leq \lambda$. Moreover when this holds, if $L$ is an irreducible subquotient of $M$, then $L$ is also an irreducible subquotient of $N$.

For (1), note that by the explicit formulas for $f_\alpha(z)$ and $h_i(z)$-actions on $W^+_{0,\kappa}$ in (2.7) and (2.9), respectively, the lexicographically ordered monomials

$$
\prod_{\ell_\alpha > 0} b_{i_\alpha, \ell_\alpha} \prod_{m_\alpha > 0} f_{\alpha_0, m_\alpha} \prod_{n_\alpha < 0} a^{\ast}_{\beta_\kappa, \kappa_\ell} \mid \beta \rangle \mid 0 \rangle^\prime
$$

form a basis of $W^+_{0,\kappa}$. The weights appearing in the coinvariants must be of the form

$$
\mu' = -\sum_j (n_j \delta - \beta_j)
$$

where $n_j > 0$ and $\beta_j \in \Delta_+$. Indeed, by the description of the basis of $W^+_{0,\kappa}$ in (2.21), there is an isomorphism of $\mathfrak{b}$-modules

$$(W^+_{0,\kappa})_{\kappa - [t^{-1}, \delta t^{-1}]} \cong \mathbb{C}[a^\ast_{\alpha_n}]_{\alpha \in \Delta_+, n < 0},$$

and $(W^+_{0,\kappa})_{\mathfrak{b}_-}$ is a quotient.

For (2) note that [KK79, Theorem 2] (also see [Fre07, §6.3.3]) gives a characterization of possible irreducible subquotient of Verma modules:

Proposition 2.23. A weight $\mu' = (\mu, \kappa)$ appears as the highest weight of an irreducible subquotient of $M_{(\lambda, 0), \kappa}$ if and only if $n \leq 0$ and $\mu = w(\rho) - \rho$ for some $w \in W$.

Note that for any $w \in W$ the weight $w(\rho) - \rho$ equals the linear combination of simple roots of $\mathfrak{g}$ with non-positive coefficients, hence the weight of any irreducible subquotient of $M_{0,\kappa}$ has the form

$$
\mu' = -n \delta - \sum_i m_i \alpha_i
$$

for some $n \geq 0$ and $m_i \geq 0$. Finally, note that (2.22) and (2.24) cannot simultaneously hold, a contradiction, and hence $V = 0$. We have thus completed (a), (b), and (e), which together prove that $M_{0,\kappa} \simeq W^+_{0,\kappa}$.

Next, we characterize all the endomorphisms of our module $M_{0,\kappa} \simeq W^+_{0,\kappa}$. In other words, we hope to characterize all $\mathfrak{g}'_\kappa$-homomorphisms $M_{0,\kappa} \to W^+_{0,\kappa}$. By adjunction, this is equivalent to characterize the vectors in $W^+_{0,\kappa}$ annihilated by $\mathfrak{b}_+$.

Lemma 2.25 ([Fre07, Lemma 6.3.4]). The space of $\mathfrak{b}_+$-invariants of $W^+_{0,\kappa}$ is equal to $\pi_{-2\rho} \subset W^+_{0,\kappa}$. 


Proof. The formulas in Theorem 2.5 shows the vectors of $\pi_{-2\rho}$ are annihilated by $\tilde{b}_+$. To prove the converse, note that $W_{0,\kappa}^+$ has another basis
\[
\prod_{\ell_{\alpha} < 0} b_{i_{\alpha},\ell_{\alpha}} \prod_{m_b \leq 0} f_{\alpha, m_b}^R \prod_{n_c < 0} a_{\alpha, n_c}^* |0\rangle',
\]
by the same argument as for (2.21). Here the $f_{\alpha, n}^R$ generate an action of $t_n [t]$ as defined in [Los24a], which we now briefly recall. There is an isomorphism of the Fock representation of $\hat{\Gamma}_\kappa$ with the vertex algebra of chiral differential operators on $N_-$:
\[
M_\kappa \simeq \text{CDO}(N_-).
\]
Now viewing $J_n = n_- [t]$ as the right-invariant vector fields on $J N_-$ defines a left $n_- [t]$-action on $\text{CDO}(N_-)$, which induces the restriction of the $\tilde{\mathfrak{g}}'_\kappa$-action on $W^+_{0,\kappa}$. On the other hand, viewing $n_- [t]$ as the left-invariant vector fields on $J N_-$ defines a right $n_- [t]$-action which are the $f_{\alpha, n}^R$.

Thus there is a tensor product decomposition
\[
W^+_{0,\kappa} = W^+_{0,\kappa} \otimes W^+_{0,\kappa},
\]
where $W^+_{0,\kappa}$ (resp., $W^+_{0,\kappa}$) is the span of monomials in $a^*_{\alpha, n}$ (resp., in $b_{i,\ell}$ and $f_{\alpha, m}^R$). Since the left action of $t_n [t]$ commutes with $b_{i,\ell}$ and $f_{\alpha, m}^R$, we conclude $t_n [t]$ acts by zero on $W^+_{0,\kappa}$. In fact, it is isomorphic to the restricted dual of the free $U(n_- [t])$-module with one generator. Thus
\[
(W^+_{0,\kappa})^{t_n [t]} = W^+_{0,\kappa} \otimes (W^+_{0,\kappa})^{t_n [t]} = W^+_{0,\kappa}.
\]
Furthermore, for $h \in \mathfrak{h}$ since
\[
[h, a^*_{\alpha, n}] = \alpha(h) a^*_{\alpha, n},
\]
a vector in $W^+_{0,\kappa}$ is annihilated by $\mathfrak{h}$ if and only if it belongs to $\pi_{-2\rho}$. 

3. PROOF OF THE KAC-KAZHDAN CONJECTURE

The Verma module $M_{\lambda,\kappa}$ over $\mathfrak{g}'_\kappa$ has a unique irreducible quotient $L_{\lambda,\kappa}$. The Kac-Kazhdan conjecture computes the character of $M_{\lambda,\kappa}$ for generic $\lambda'$.

First, recall that the roots $\tilde{\Delta}_+$ from (2.11) has a subset of real roots
\[
\tilde{\Delta}^\text{re}_+ := (\Delta_+ + \mathbb{Z}_{\geq 0} \delta) \cup (\Delta_- + \mathbb{Z}_{> 0} \delta),
\]
i.e., the roots $(\lambda, \phi) \in \tilde{\Delta}_+$ such that $\lambda \neq 0$.

Theorem 3.1. For a generic weight $\lambda \in \mathfrak{h}^*$ of critical level,
\[
\text{ch} L_{\lambda,\kappa} = e^{\lambda'} \prod_{\alpha' \in \Delta^\text{re}_+} (1 - e^{-\alpha'})^{-1}.
\]
Here, a weight $\lambda$ is generic when $\lambda \notin \bigcup_{\beta \in \tilde{\Delta}^\text{re}_+, m > 0} H_{\beta, m}^\text{nc}$ where $H_{\beta, m}^\text{nc}$ are certain hyperplanes in $\mathfrak{h}^*$ defined in [KK79].

Proof. For $\lambda \in \mathfrak{h}^*$ the Wakimoto module of critical level $W_{\lambda/t}$ is a $\mathfrak{g}'_{\kappa_c}$-module since $M_\kappa$ is graded, and the $\mathfrak{h}(t)$-module $C_{\lambda/t}$ is graded, and hence $W_{\lambda/t} := M_\kappa \otimes C_{\lambda/t}$ inherits a grading. The $\mathfrak{g}'_{\kappa_c}$-module $W_{\lambda/t}$ has character
\[
\text{ch} W_{\lambda/t} = e^{\lambda'} \prod_{\alpha' \in \Delta^\text{re}_+} (1 - e^{-\alpha'})^{-1},
\]
where $\lambda' = (\lambda, 0)$, from a similar argument as in (a) in the proof of Theorem 2.3. Moreover, the same argument as in (b) in the proof of Theorem 2.3 shows there is a homomorphism $M_{\lambda',\kappa_c} \to W_{\lambda/t}$.

Proof.
sending the highest weight vector to $|0\rangle$. It thus suffices to check that if $\lambda$ is a generic weight of critical level, then $W_{\lambda/t}$ is irreducible, since then $L_{\lambda',\kappa_c} \simeq W_{\lambda/t}$. If $W_{\lambda/t}$ is not irreducible, either:

- $W_{\lambda/t}$ is not generated by its highest vector, i.e., the homomorphism $\mathbb{M}_{\lambda',\kappa_c} \to W_{\lambda/t}$ is not surjective;
- or $W_{\lambda/t}$ is generated by its highest vector, in which case $L_{\lambda',\kappa_c} \simeq W_{\lambda/t}$.

If $W_{\lambda/t}$ is not irreducible, either:

- $W_{\lambda/t}$ is not generated by its highest vector, i.e., the homomorphism $\mathbb{M}_{\lambda',\kappa_c} \to W_{\lambda/t}$ is not surjective;
- or $W_{\lambda/t}$ is generated by its highest vector, in which case $L_{\lambda',\kappa_c} \simeq W_{\lambda/t}$.

Thus, $W_{\lambda/t}$ contains a singular vector not in $\mathbb{C}|0\rangle$ then it must be annihilated by $n_+ [t]$. We know that

$$\prod_{n_a<0} e^{R}_{\alpha_a,n_a} \prod_{m_b \leq 0} a^*_{\alpha_b,m_b} |0\rangle$$

forms a basis of $M_{\rho}$, where the $e^{R}_{\alpha_a,n_a}$ are defined as in the proof of Lemma 2.25, using the description of $M_{\rho} \simeq \text{CDO}(N_+)$, as in the proof of Lemma 2.25. By the same method as in Lemma 2.25, the $n_+ [t]$-invariants of $W_{\lambda',\kappa_c}$ equals the subspace $W_{\lambda',\kappa_c}$ spanned by all monomials of $e^{R}_{\alpha_a,n_a}$. In particular, the weight of any singular vector of $W_{\lambda/t}$ is of the form $\lambda' - \sum_j (n_j \delta - \beta_j)$ where $n_j > 0$ and $\beta_j \in \Delta_+$. Thus $W_{\lambda/t}$ contains an irreducible subquotient of that highest weight. Now, since for $\alpha' \in \Delta_+$,

$$\text{mult} \alpha' = \begin{cases} 1 & \text{if } \alpha' \in \Delta_+^{	ext{re}} \\ \ell & \text{otherwise,} \end{cases}$$

we have

$$\text{ch} \mathbb{M}_{\lambda',\kappa_c} = \prod_{n>0} (1 - e^{-n\delta})^{-\ell} \text{ch} W_{\lambda/t},$$

where $\ell$ is the rank of $\mathfrak{g}$. Thus if an irreducible module $L_{\mu',\kappa_c}$ appears as a subquotient of $W_{\lambda/t}$, it must also appear as a subquotient of $\text{ch} \mathbb{M}_{\lambda',\kappa_c}$: only look at the part of (3.2) with energy zero. But our contradicts the assumption that $\lambda$ is generic: irreducible subquotients of Verma modules are controlled by hyperplanes by [KK79]. Thus $W_{\lambda/t}$ does not contain any singular vectors other than the highest weight.

Next, if $W_{\lambda/t}$ is not generated by its highest vector, then by the same argument as above there is an irreducible subquotient of $W_{\lambda/t}$ with highest weight $\lambda' - \sum_j (n_j \delta + \beta_j)$ with $n_j \geq 0$ and $\beta_j \in \Delta_+$. This again contradicts $\lambda$ being a generic weight. \qed

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