Why should we expect the Kazhdan-Lusztig conjecture?

1.0) Recap & goals

Let $\lambda \in \Lambda_+$ and $S := W \cdot \lambda$. Consider the infinitesimal block $O^X$. The indecomposable projectives are $P(W \cdot \lambda)$'s from Lec 24. Set $P := \bigoplus_{w \in W} P(w \cdot \lambda)$, $A := \text{End}(P)$. Note that $P$ is a projective generator, so $O^X \cong A$-$\text{mod}$, see Sec 1.5 of Lec 23.

It turns that the KL conjecture morally follows from the existence of a "positive algebra grading" on $A$ (the meaning of this will be explained below). To make an actual argument one needs a bit more structure. We will elaborate on this in the present lecture.

1.1) $K_0$-group.

We define the Grothendieck group $K_0(O^X)$ as the group generated by symbols $[M]$ for isomorphism classes $M$ of objects in $O^X$ and relations of the form $[M] = [M'] + [M'']$ for all SES $0 \to M' \to M \to M'' \to 0$.

Since every object in $O^X$ has a JH filtration, $K_0(O^X)$ is a free abelian group w. basis $[L]$, $L \in \text{Irr}(O^X)$.

Exercise: Let $F : O^X \to O^X$ be an exact functor. Then $[F] : [M] \mapsto [FM]$ extends to a unique group endomorphism of $K_0(O^X)$. 

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Now we give a description of $K_0(O^x)$. 

Proposition: 1) The elements $[\Delta(w;\lambda)]$, $w \in W$, form a basis of $K_0(O^x)$. 

2) Under the identification, $K_0(O^x) \cong \mathbb{Z}W$, $[\Delta(w;\lambda)] \mapsto w$, the operator $[\Theta_i]$ on $K_0(O^x)$ becomes the operator of right multiplication by $s_i + 1$, $i = 1, \ldots, n-1$.

Proof: 1) holds because the $([\Delta(w;\lambda)])_{w \in W}$ is obtained from $([L(w;\lambda)])$ by applying a uni-triangular matrix. 2) follows from the claim that $\Theta_i \Delta(w;\lambda)$ fits as the second term into an SES $w_2$ other terms $\Delta(w;\lambda)$ & $\Delta(w_2;\lambda)$, Proposition in Sec 1.2 of Lec 23.

Now we are going to relate $K_0(O^x)$ and the split $K_0$-group $K_0(O^x_{\text{proj}})$, see Section 1.3 & (v) of Sec 1.4 of Lec 26 for a discussion of the latter.

We have a map $K_0(O^x_{\text{proj}}) \to K_0(O^x)$ sending $[P(w;\lambda)]$ to $[P(w;\lambda)]$. This map is an isomorphism: the transition matrix from $[\Delta(w;\lambda)]$s to $[P(w;\lambda)]$s is uni-triangular by Thm in Sec 1.2 of Lec 24.

We also have a pairing $K_0(O^x_{\text{proj}}) \times K_0(O^x) \to \mathbb{Z}$ given by

$\left< [P], [M] \right> = \dim \text{Hom}_{O^x}(P, M)$

Exercise: 1) Prove that this is a well-defined pairing and $[P(w;\lambda)]$s & $[L(w;\lambda)]$s are dual bases.
We can carry the pairing to \( K_0(\mathcal{O}^X) \times K_0(\mathcal{O}^Y) \rightarrow \mathbb{Z} \) by using the isomorphism \( K_0(\mathcal{O}^X \text{proj}) \cong K_0(\mathcal{O}^X) \). Prove that the classes \( \{\Delta(w\cdot \lambda)\} \), \( w \in W \), form an orthonormal basis.

12) Graded \( K_0 \).

The problems of computing the characters of irreducibles/their multiplicities in the Verma's reduces to expressing the basis \( \{\xi(w\cdot \lambda)\} \) via the standard basis in \( K_0(\mathcal{O}^X) = \mathbb{Z} W \). Recall, Section 1.2 of Lecture 19, that \( \mathbb{Z} W \cong \mathcal{H}_W^\mathcal{V}(w) / (t-1) = \mathcal{H}_W^\mathcal{V}(w) / (\sigma - \varepsilon) \), \( \varepsilon \in \{\pm 1\} \).

Let's recall one of the parts of KL conjecture (Sec 1.4 of Lec 21). Identify \( K_0(\mathcal{O}^X) \) w. \( \mathbb{Z} W \) by sending \( \{\Delta(w\cdot \lambda)\} = \{\xi(w\cdot \lambda)\} \) to \( \xi \).

Then what we need to show is \( \{\xi(w\cdot \lambda)\} = C_w|_{\sigma = 1} \).

There's no known way to characterize \( C_w|_{\sigma = 1} \) combinatorially. So we should ask if there's an "upgrade" of \( \mathcal{O}^X \) so that its \( K_0 \) is \( \mathcal{H}_W^\mathcal{V}(w) \) and the basis of irreducibles is \( C_w \), \( w \in W \) (w some twist). This upgrade comes from introducing a grading on \( A = \text{End}(\mathcal{O})^{op} \) and considering the category of graded \( A \)-modules.

Before we discuss how to introduce the grading on \( A \) let's discuss "graded \( K_0 \)'s" - compare to Sec 1.3 in Lec 26.

Let \( \mathcal{R} \) be a \( \mathbb{Z} \)-graded finite dimensional \( C \)-algebra. As in Sec 1.2 of Lec 25, it makes sense to speak about the category \( \mathcal{R} \)-gmod of graded finite dimensional \( \mathcal{R} \)-modules. It comes w grading shift endo-functors \( \langle ? \rangle \).
We can talk about $K_0(R\text{-grmod})$ just as before, using graded module homomorphisms in SES's. The exact functor $<j>$ defines an endomorphism $[<j>]$ of $K_0(R\text{-grmod})$. We equip $K_0(R\text{-grmod})$ with a $\mathbb{Z}[v^{\pm 1}]$-module structure by making $v$ act by $[<-1>]$. 

Example: The irreducible objects in $C\text{-grmod}$ are exactly $C<j>$ with $j \in \mathbb{Z}$. This is a 1-dimensional vector space in degree $j$. So $K_0(C\text{-grmod}) \cong \mathbb{Z}[v^{\pm 1}]$, the regular $\mathbb{Z}[v^{\pm 1}]$, where $v \in \mathbb{Z}[v^{\pm 1}]$ corresponds to $C$ in degree 0.

Now we want to compare $K_0(R\text{-grmod})$ with $K_0(R\text{-mod})$. We have the functor of forgetting the grading $R\text{-grmod} \rightarrow R\text{-mod}$, which gives rise to $K_0(R\text{-grmod}) \rightarrow K_0(R\text{-mod})$ that factors through

$$K_0(R\text{-grmod})/(v-1)K_0(R\text{-grmod}) \rightarrow K_0(R\text{-mod}) \quad (1)$$

Fact: (to be elaborated below in a special case; compare to Sec 2.2 in Lec 25). Every irreducible $R$-module admits a grading, unique up a shift, meaning that if $L,L'' \in R\text{-grmod}$ are both isomorphic to $L \in \text{Irr}(R)$ as $R$-modules, then $\exists! \, j \in \mathbb{Z}$ with $L \cong L''<j>$ in $R\text{-grmod}$.

Corollary: $K_0(R\text{-grmod})$ is a free $\mathbb{Z}[v^{\pm 1}]$-module & (1) is an isomorphism.

1.3) Grading on $A$. 

Recall that $A$ stands for $\text{End}(\oplus_{\omega \in \mathbb{N}} P(\omega))^{op}$, where the endomorphism
algebra is taken in the category $O^\text{X-proj}$. Recall an equivalence $O^\text{X-proj} \rightarrow SMod_{\text{ungr}}$, Sec. 1.4 on Lec 25. So $A = \text{End}_{SMod_{\text{ungr}}} (\oplus B_w)$. The $\mathbb{C}[v^\pm]$-module $B_w$ is graded. So the endomorphism algebra $A$ acquires a grading. We claim that $K_0(A\text{-grmod})$ is identified with $H_v(W)$.

Let $A\text{-grproj}$ denote the category of projective objects in $A\text{-grmod}$. ($\equiv P \in A\text{-grmod}$, projective as $A$-modules). Every indecomposable in $A\text{-grproj}$ is indecomposable in $A\text{-proj}$; conversely, every indecomposable in $A\text{-proj}$ admits a unique grading (up to shift and isomorphism).

Note that $SMod \rightarrow A\text{-grproj}$ via $B \mapsto \text{Hom}(\oplus B_w, B)$ (exercise). So $K_0(A\text{-grproj}) \rightarrow K_0(SMod) = H_v(W)$ (see (v) in Sec 1.4 of Lec 26).

The groups $K_0(A\text{-grproj})$ & $K_0(A\text{-grmod})$ have natural $\mathbb{Z}[v^{\pm1}]$-module structures. We still have a perfect pairing

$$K_0(A\text{-grproj}) \times K_0(A\text{-grmod}) \rightarrow \mathbb{Z}[v^{\pm1}], \quad \langle [P], [M] \rangle = [\text{Hom}_A(P, M)],$$

where the r.h.s. is in $K_0(C\text{-grmod}) = \mathbb{Z}[v^{\pm1}]$

**Exercise:** Show that this pairing is $\mathbb{Z}[v^{\pm1}]$-linear in the 2nd argument and $\mathbb{Z}[v^{\pm1}]$-semilinear (w.r.t $\delta$) in the 1st argument.

Recall, Secs 1.3, 1.4 of Lec 26, that $K_0(A\text{-grproj}) = K_0(SMod)$ is identified with $H_v(W)$. So $K_0(A\text{-grmod})$ gets identified with its semilinear dual. One can still think about this module as $H_v(W)$ - it’s still a free $\mathbb{Z}[v^{\pm1}]$-module w. basis indexed by $W$. We declare that the dual basis vector of $H_w \in K_0(A\text{-grproj}) = H_v(W)$ is $H_w v^0$. 

Example: Let $n=2$. Then the basis of indecomposables in $K_0(A\text{-}\text{proj})$
$= H_0(W)$ is $H_4, H_5 + v$. The dual basis of this is $H_5 - v^{-1}H_4, H_4$. Indeed,
$\langle H_4, H_5 - v^{-1}H_4 \rangle = \langle H_4, H_5 \rangle = 1, \quad \langle H_5 + v, H_5 - v^{-1}H_4 \rangle = v^{-1} - v^{-1} = 0$.

Let’s explain a representation theoretic interpretation of $H_w \in K_0(A\text{-}\text{grmod})$
under this identification. A general expectation is that all “nice” objects
in $O^X \to A$-mod admit gradings. If the object in question is indecompos-
able, then the corresponding grading is unique (up to a shift and isomorphism).
It turns out that $H_w \in K_0(A\text{-}\text{grmod})$ is the $K_0$-class of a graded $A$-module corresponding to $\mathcal{O}(ww_0 \lambda) \in O^X$.

A reason for this choice comes from the behavior of the functors
$\mathcal{O}_i : O^X \to O^X$ and their “graded lifts” to $A\text{-}\text{grmod}$. We won’t
elaborate on this, let’s just point out that this choice works
nicely for our purposes.

Now we proceed to an important result in the Soergel theory that
can be used to explain the KL conjecture.

Fact: The grading on $A$ is positive meaning that $A_i = 0$ for $i < 0$ and
$\text{Rad } A = \bigoplus_{i > 0} A_i$.

Note that under these conditions we can choose a preferred grading on
every irreducible $A$-module: just put it in degree $0$.

Remark: It turns out, also a result of Soergel, that one can establish
a graded algebra isomorphism between $A$ and another graded algebra,
where the grading is manifestly positive. The algebra in question is the Ext-algebra $\text{Ext}^0_{\mathcal{O}} \left( \bigoplus_{w \in W} L(w, \lambda) \right)$ (for the algebra structure on $\text{Ext}$, see A.3.1.1 in Eisenbud's "Commutative algebra...". An isomorphism $\text{End}_{\mathcal{O}} \left( \bigoplus_{w \in W} P(w, \lambda) \right) \cong \text{Ext}^0_{\mathcal{O}} \left( \bigoplus_{w \in W} L(w, \lambda) \right)$ is a part of a bigger picture - the Kostul duality, see A. Beilinson, V. Ginzburg, W. Soergel "Kostul duality patterns in Representation theory", J. Amer. Math. Soc. (1996), n.2, 473-527.

1.4) Relevance to KL conjecture

Let's explain how the positivity of grading (and the existence of a nice duality on $A$-grmod KL conjectures).

The KL basis is characterized by two properties (see Sec. 1.2 of Lec 21), although now we use $v^{-1}$ instead of $v$.

(i) $C_w \subseteq H_w + v^{-1}\text{Span}_{\mathbb{Z}[v^{-1}]}(H_w) \iff H_w \subseteq C_w + v^{-1}\text{Span}_{\mathbb{Z}[v^{-1}]}(C_w), \forall w \in W$

(ii) $C_w = \overline{C_w}, \forall w \in W.$

For example, for $n=2$, we have $C_4 = H_4, C_5 = H_5 - v^{-1}H_4$ - compare to Example in Sec 1.3.

Condition (i) turns out to be about the positivity of the grading.

**Lemma:** Suppose $A$ is a positively graded finite dimensional algebra. Let $L$ be an irreducible $A$-module in deg 0. Let $M \in A$-grmod be such that $L$ is its unique irreducible submodule. Then $(M/L)_0 = 0$.
for all $i \geq 0$, equivalently, $M/\mathfrak{m}$ is filtered by irreducible $A$-modules in negative degrees.

**Proof:** Let $N \in A$-$\text{grmod}$, let $i \in \mathbb{Z}$ be such that $N_j = 0 \forall j \neq i$. Then every $A_i$-submodule in $N_i$ is also an $A$-submodule ($4c\ A_i = 0$ for $i < 0$). It follows that $M_j = 0 \forall j \geq 0$ and, since $A_i$ is semisimple, that $M_0 = L$. This is equivalent to the claim of the lemma. $\square$

We apply this observation as follows. Take $L \in \text{Irr}(A)$ in deg 0 that corresponds to $L(ww, \lambda) \in \text{Irr}(A^x)$. Take $M$ to be a graded module that corresponds to $\nu(ww, \lambda)$. As an ordinary module, its unique irreducible submodule is $L$. So it's also its unique graded submodule (up to a shift). We can shift the grading on $M$ and assume that $L$ in deg 0 is the unique irreducible submodule. Then Lemma says that in $K_0(A$-$\text{grmod})$ we have

$$[M] \in [L] + \nu^{-1}\text{Span}_{\mathbb{Z}[\nu^{-1}]}([L'])$$

This is (i).

Now we proceed to (ii). This requires the examination of duality. We note that if $M$ is an $A$-module, $M^*$ is an $A^{op}$-module. If $M$ is graded, $M = \bigoplus_{i \in \mathbb{Z}} M_i$, then $M_i^*$ is graded: $(M_i^*)_i = (M_i^*)_i$ (so that the pairing map $M \otimes M^* \to \mathbb{C}$ has degree 0). So we get a contravariant equivalence $A$-$\text{grmod} \to A^{op}$-$\text{grmod}$. If we have a graded algebra isomorphism, say $\phi: A \to A^{op}$, we get a contravariant self-equivalence of $A$-$\text{grmod}$ $(M \mapsto M^*$ with action twisted by $\phi$) to be denoted by $\text{ID}_\phi$. 

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Exercise: $[\tilde{D}_p]: K_0(A\text{-grmod}) \rightarrow K_0(A\text{-grmod})$ is $\mathbb{Z}[v^\pm]$-semi-linear

(w.r.t. $v \mapsto v^{-1}$)

$\tilde{D}_p$ sends an irreducible in deg 0 to an irreducible in deg 0. Now suppose that $\tilde{D}_p$ fixes all irreps. Then the basis $[L]$ of $K_0(A\text{-grmod})$ is fixed by $[\tilde{D}_p]$.

Let's explain how the identification $A \rightarrow A^{op}$ works. We have $A = \text{End}_R(\bigoplus_{w \in W} B_w)^{op}$, where $R = \mathbb{C}[y_1, \ldots, y_n]$. This realization gives an identification $A \rightarrow A^{op}$ of graded algebras. First of all, note that $A^{op} = \text{End}_R(\bigoplus_{w \in W} B_w^*)$, where $B_w^*$ is dual graded module.

Now to identify $A$ with $A^{op}$ it's enough to establish a graded $R$-module isomorphism $B_w \cong B_w^*$. The corresponding equivalence $\tilde{D}_p$ is forced to fix each deg 0 irreducible $A$-module (exercise).

To show that $B_w \cong B_w^*$ we can argue as follows.

Lemma: We have a graded $R$-module isomorphism $BS_w \cong BS_w^*$

Proof: It's enough prove that there's a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $BS_w$ (this yields a vector space isomorphism $BS_w \rightarrow BS_w^*$) s.t. $(r_b, r_{b'}) = (r, r_{b'})$ (this condition implies that the isomorphism $BS_w \rightarrow BS_w^*$ is $R$-linear), and the degree of $(\cdot, \cdot)$ is 0 (so that $BS_w \rightarrow BS_w^*$ is graded). To prove the existence of a form we can argue inductively: suppose $M \in R$-grmod has a degree 0 $R$-invariant symmetric bilinear form $(\cdot, \cdot)_M$. We define a form $(\cdot, \cdot)$ on $R \otimes_{R_s} M$ as follows.

Recall, Section 1.1 of Lec 25, that $R \otimes_{R_s} M \cong_1 M + R \otimes_{R_s} M$. We define
(\cdot, \cdot) on R@\rho M by setting:

\[(h_5 \otimes m, h_5 \otimes m') = (h_5 \otimes m, h_5 \otimes m') = 0\]
\[(h_5 \otimes m, h_5 \otimes m') = (h_5 \otimes m, h_5 \otimes m') = (m, m')_M.\]

It's an exercise to check that (\cdot, \cdot) has required properties. □

**Exercise:** Use BS\_w \sim BS\_w^* and Theorem in Sec 1.2 of Lec 26 (and the Krull-Schmidt theorem) to check that Bw \sim B^*_w.

**Conclusion:**

So, we have checked the properties (i) and (ii). This shows that under the identification of K(A-gmod) \sim H_w(w) where the graded A-module corresponding to \n(w, \lambda) \in O^x is sent to H_w the classes of deg 0 irreducible modules are the KL basis in the normalization above. To prove this we need to know that A is positively graded and there's an isomorphism \nu: A \rightarrow A^{opp} of graded algebras s.t. the resulting contravariant self-equivalence \text{Def}_\nu of A-mod fixes each irreducible.