Why should we expect the Kazhdan-Lusztig conjecture?

1.0) Recap & goals

Let $\lambda \in \Lambda_+$ and $\chi := W\cdot\lambda$. Consider the infinitesimal block $\mathcal{O}^\chi$. The indecomposable projectives are $P(w\cdot\lambda)$'s from Lec 24. Set $P := \bigoplus_{w\in W} P(w\cdot\lambda)$, $A := \mathrm{End}(P)^{opp}$. Note that P is a projective generator, so $\mathcal{O}^\chi \xrightarrow{\sim} A\text{-mod}$, see Sec 1.5 of Lec 23.

It turns that the KL conjecture morally follows from the existence of a "positive algebra grading" on A (the meaning of this will be explained below). To make an actual argument one needs a bit more structure. We will elaborate on this in the present lecture.

1.1) K_0-group.

We define the Grothendieck group $K_0(\mathcal{O}^\chi)$ as the group generated by symbols $[M]$ for isomorphism classes M of objects in $\mathcal{O}^\chi$ and relations of the form $[M]=[M']+[M'']$ for all SES

$$0\to M'\to M\to M''\to 0.$$

Since every object in $\mathcal{O}^\chi$ has a JH filtration, $K_0(\mathcal{O}^\chi)$ is a free abelian group w. basis $[L]$, $L\in \mathrm{Irr}(\mathcal{O}^\chi)$.

Exercise: Let $\mathcal{F}:\mathcal{O}^\chi\to\mathcal{O}^\chi$ be an exact functor. Then $[\mathcal{F}]:[M]\mapsto[\mathcal{F}M]$ extends to a unique group endomorphism of $K_0(\mathcal{O}^\chi)$.

Now we give a description of $K_0(\mathcal{O}^\chi)$.

Proposition: 1) The elements $[\Delta(w\cdot\lambda)], w\in W$, form a basis of $K_0(\mathcal{O}^\chi)$.
2) Under the identification, $K_0(\mathcal{O}^\chi)\xrightarrow{\sim}\mathbb{Z}W$, $[\Delta(w\cdot\lambda)]\mapsto w$, the operator $[\Theta_i]$ on $K_0(\mathcal{O}^\chi)$ becomes the operator of right multiplication by s_i+1, $i=1,\dots n-1$.

Proof: 1) holds b/c the $([\Delta(w\cdot\lambda)])_{w\in W}$ is obtained from $([L(w\cdot\lambda)])$ by applying a uni-triangular matrix. 2) follows from the claim that $\Theta_i\Delta(w\cdot\lambda)$ fits as the second term into an SES w. 2 other terms $\Delta(w\cdot\lambda)$ & $\Delta(ws_i\cdot\lambda)$, Proposition in Sec 1.2 of Lec 23. $\square$

Now we are going to relate $K_0(\mathcal{O}^\chi)$ and the split K_0-group $K_0(\mathcal{O}^\chi\text{-proj})$, see Section 1.3 & (v) of Sec 1.4 of Lec 26 for a discussion of the latter.

We have a map $K_0(\mathcal{O}^\chi\text{-proj})\longrightarrow K_0(\mathcal{O}^\chi)$ sending $[P(w\cdot\lambda)]$ to $[P(w\cdot\lambda)]$. This map is an isomorphism: the transition matrix from $[\Delta(u\cdot\lambda)]$'s to $[P(w\cdot\lambda)]$'s is uni-triangular by Thm in Sec 1.2 of Lec 24.

We also have a pairing $K_0(\mathcal{O}^\chi\text{-proj})\times K_0(\mathcal{O}^\chi)\longrightarrow\mathbb{Z}$ given by $\langle [P],[M]\rangle := \dim \operatorname{Hom}_{\mathcal{O}^\chi}(P,M)$

Exercise: 1) Prove that this is a well-defined pairing and $[P(w\cdot\lambda)]$'s & $[L(w\cdot\lambda)]$'s are dual bases.

2) We can carry the pairing to $K_0(\mathcal{O}^\chi)\times K_0(\mathcal{O}^\chi)\to\mathbb{Z}$ by using the isomorphism $K_0(\mathcal{O}^\chi\text{-proj})\xrightarrow{\sim}K_0(\mathcal{O}^\chi)$. Prove that the classes $[\Delta(w\cdot\lambda)]$, $w\in W$, form an orthonormal basis.

1.2) Graded K_0.

The problems of computing the characters of irreducibles/their multiplicities in the Vermas reduces to expressing the basis $[L(w\cdot\lambda)]$ via the standard basis in $K_0(\mathcal{O}^\chi)=\mathbb{Z}W$. Recall, Section 1.2 of Lecture 19, that $\mathbb{Z}W\xrightarrow{\sim}\mathcal{H}^{\mathbb{Z}}(W)/(t-1)=\mathcal{H}_v(W)/(v-\varepsilon)$, $\varepsilon\in\{\pm1\}$.

Let's recall one of the parts of KL conjecture (Sec 1.4 of Lec 21) Identify $K_0(\mathcal{O}^\chi)$ w. $\mathbb{Z}W$ by sending $[\Delta(w\cdot\lambda^-)](=\Delta(ww_0\cdot\lambda))$ to w. Then what we need to show is $[L(w\cdot\lambda^-)]=C_w|_{v=-1}$.

There's no known way to characterize $C_w|_{v=-1}$ combinatorially. So we should ask if there's an "upgrade" of $\mathcal{O}^\chi$ so that it's K_0 is $\mathcal{H}_v(W)$ and the basis of irreducibles is C_w, $w\in W$ (w. some twist). This upgrade comes from introducing a grading on $A=End(P)^{opp}$ and considering the category of graded A-modules.

Before we discuss how to introduce the grading on A let's discuss "graded K_0's" - compare to Sec 1.3 in Lec 26.

Let R be a $\mathbb{Z}$-graded finite dimensional $\mathbb{C}$-algebra. As in Sec 1.2 of Lec 25, it makes sense to speak about the category R-grmod of graded finite dimensional R-modules. It comes w. grading shift endofunctors $\langle ?\rangle$.

We can talk about $K_0(R\text{-grmod})$ just as before, using graded module homomorphisms in SES's). The exact functor $\langle j\rangle$ defines an endomorphism $[\langle j\rangle]$ of $K_0(R\text{-grmod})$. We equip $K_0(R\text{-grmod})$ with a $\mathbb{Z}[v^{\pm 1}]$-module structure by making v act by $[\langle -1\rangle]$.

Example: The irreducible objects in $\mathbb{C}$-grmod are exactly $\mathbb{C}\langle j\rangle$ w. $j\in\mathbb{Z}$. This is a 1-dimensional vector space in deg $-j$. So $K_0(\mathbb{C}\text{-grmod})\xrightarrow{\sim}\mathbb{Z}[v^{\pm 1}]$, the regular $\mathbb{Z}[v^{\pm 1}]$, where $1\in\mathbb{Z}[v^{\pm 1}]$ corresponds to $\mathbb{C}$ in degree 0.

Now we want to compare $K_0(R\text{-grmod})$ w. $K_0(R\text{-mod})$. We have the functor of forgetting the grading $R\text{-grmod}\longrightarrow R\text{-mod}$. It gives rise to $K_0(R\text{-grmod})\to K_0(R\text{-mod})$ that factors through

$$K_0(R\text{-grmod})/(v-1)K_0(R\text{-grmod})\longrightarrow K_0(R\text{-mod}) \qquad (1)$$

Fact (to be elaborated below in a special case; compare to Sec 2.2 in Lec 25). Every irreducible R-module admits a grading, unique up a shift, meaning that if $L', L''\in R\text{-grmod}$ are both isomorphic to $L\in \operatorname{Irr}(R)$ as R-modules, then $\exists!\, j\in\mathbb{Z}$ w $L'\simeq L''\langle j\rangle$ in R-grmod.

Corollary: $K_0(R\text{-grmod})$ is a free $\mathbb{Z}[v^{\pm 1}]$-module & (1) is an isomorphism.

1.3) Grading on A.

Recall that A stands for $\operatorname{End}(\bigoplus_{w\in W} P(w\cdot\lambda))^{opp}$, where the endomorphism algebra is taken in the category $\mathcal{O}^\chi$-proj. Recall an equivalence $\mathcal{O}^\chi\text{-proj} \xrightarrow{\sim} SMod_{ungr}$, Sec. 1.4 on Lec 25. So $A = \operatorname{End}_{SMod_{ungr}}(\bigoplus \underline{B}_w)^{opp}$. The $\mathbb{C}[\mathfrak{h}^*]$-module $\underline{B}_w$ is graded. So the endomorphism algebra A acquires a grading. We claim that $K_0(A\text{-grmod})$ is identified w. $\mathcal{H}_v(W)$.

Let A-grproj denote the category of projective objects in A-grmod. ($\Leftrightarrow P \in A$-grmod, projective as A-modules). Every indecomposable in A-grproj is indecomposable in A-proj, conversely every indecomposable in A-proj admits a unique grading (up to shift and isomorphism).

Note that $SMod \xrightarrow{\sim} A\text{-grproj}$ via $\underline{B} \mapsto \operatorname{Hom}(\bigoplus_w \underline{B}_w, \underline{B})$ (exercise). So $K_0(A\text{-grproj}) \xrightarrow{\sim} K_0(SMod) = \mathcal{H}_v(W)$ (see (v) in Sec 1.4 of Lec 26).

The groups $K_0(A\text{-grproj})$ & $K_0(A\text{-grmod})$ have natural $\mathbb{Z}[v^{\pm 1}]$-module structures. We still have a perfect pairing

$$K_0(A\text{-grproj}) \times K_0(A\text{-grmod}) \longrightarrow \mathbb{Z}[v^{\pm 1}], \quad \langle [P], [M] \rangle = [\operatorname{Hom}_A(P, M)],$$

where the r.h.s. is in $K_0(\mathbb{C}\text{-grmod}) = \mathbb{Z}[v^{\pm 1}]$

Exercise: Show that this pairing is $\mathbb{Z}[v^{\pm 1}]$-linear in the 2nd argument and $\mathbb{Z}[v^{\pm 1}]$-semilinear (w.r.t $\bar{\cdot}$) in the 1st argument.

Recall, Secs 1.3, 1.4 of Lec 26, that $K_0(A\text{-grproj}) = K_0(SMod)$ is identified w. $\mathcal{H}_v(W)$. So $K_0(A\text{-grmod})$ gets identified w. its semilinear dual. One can still think about this module as $\mathcal{H}_v(W)$ – it's still a free $\mathbb{Z}[v^{\pm 1}]$-module w. basis indexed by W. We declare that the dual basis vector of $H_w \in K_0(A\text{-grproj}) = \mathcal{H}_v(W)$ is H_{ww_0}.

Example: Let $n=2$. Then the basis of indecomposables in $K_0(A\text{-grproj}) = \mathcal{H}_v(W)$ is H_1, H_s+v. The dual basis of this is $H_s - v^{-1}H_1, H_1$. Indeed,
$\langle H_1, H_s - v^{-1}H_1\rangle = \langle H_1, H_s\rangle = 1, \quad \langle H_s+v, H_s - v^{-1}H_1\rangle = v^{-1} - v^{-1} = 0.$

Let's explain a representation theoretic interpretation of $H_w \in K_0(A\text{-grmod})$ under this identification. A general expectation is that all "nice" objects in $\mathcal{O}^\chi \xrightarrow{\sim} A\text{-mod}$ admit gradings. If the object in question is indecomposable, then the corresponding grading is unique (up to a shift and isomorphism). It turns out that $H_w \in K_0(A\text{-grmod})$ is the K_0-class of a graded A-module corresponding to $\nabla(ww_0\cdot\lambda) \in \mathcal{O}^\chi$.

A reason for this choice comes from the behavior of the functors $\Theta_i : \mathcal{O}^\chi \longrightarrow \mathcal{O}^\chi$ and their "graded lifts" to A-grmod. We won't elaborate on this, let's just point out that this choice works nicely for our purposes.

Now we proceed to an important result in the Soergel theory that can be used to explain the KL conjecture.

Fact: The grading on A is positive meaning that $A_i = 0$ for $i<0$ and $\operatorname{Rad} A = \bigoplus_{i>0} A_i$.

Note that under these conditions we can choose a preferred grading on every irreducible A-module: just put it in degree 0.

Remark: It turns out, also a result of Soergel, that one can establish a graded algebra isomorphism between A and another graded algebra, where the grading is manifestly positive. The algebra in question is the Ext-algebra $\mathrm{Ext}^{\bullet}_{\mathcal{O}^x}(\bigoplus_{w\in W} L(w\cdot\lambda))$ (for the algebra structure on Ext, see A.3.11.1 in Eisenbud's "Commutative algebra..". An isomorphism $\mathrm{End}_{\mathcal{O}^x}(\oplus P(w\cdot\lambda)) \simeq \mathrm{Ext}^{\bullet}_{\mathcal{O}^x}(\oplus L(w\cdot\lambda))$ is a part of a bigger picture - the Koszul duality, see A. Beilinson, V. Ginzburg, W. Soergel "Koszul duality patterns in Representation theory", J. Amer. Math. Soc. (1996), n.2, 473-527.

1.4) Relevance to KL conjecture

Let's explain how the positivity of grading (and the existence of a nice duality on A-grmod KL conjectures).

The KL basis is characterized by two properties (see Sec. 1.2 of Lec 21), although now we use v^{-1} instead of v.

(i) $C_w \in H_w + v^{-1}\mathrm{Span}_{\mathbb{Z}[v^{-1}]}(H_u) \Leftrightarrow H_w \in C_w + v^{-1}\mathrm{Span}_{\mathbb{Z}[v^{-1}]}(C_u)$, $\forall w\in W$.

(ii) $C_w = \overline{C_w}$, $\forall\, w\in W$.

For example, for $n=2$, we have $C_1=H_1$, $C_s=H_s-v^{-1}H_1$ - compare to Example in Sec 1.3.

Condition (i) turns out to be about the positivity of the grading

Lemma: Suppose A is a positively graded finite dimensional algebra. Let L be an irreducible A-module in deg 0. Let $M\in A$-grmod be such that L is its unique irreducible submodule. Then $(M/L)_i=0$

for all $i \geq 0$, equivalently, M/L is filtered by irreducible A-modules in negative degrees.

Proof: Let $N \in A$-grmod. Let $i \in \mathbb{Z}$ be such that $N_j = 0\ \forall j > i$. Then every A_0-submodule in N_i is also an A-submodule (b/c $A_i = 0$ for $i < 0$). It follows that $M_j = 0\ \forall j > 0$ and, since A_0 is semisimple, that $M_0 = L$. This is equivalent to the claim of the Lemma. $\square$

We apply this observation as follows. Take $L \in Irr(A)$ in deg 0 that corresponds to $L(ww_0 \cdot \lambda) \in Irr(\mathcal{O}^\chi)$. Take M to be a graded module that corresponds to $\nabla(ww_0 \cdot \lambda)$. As an ordinary module, its unique irreducible submodule is L. So its also it's unique graded submodule (up to a shift). We can shift the grading on M and assume that L in deg 0 is the unique irreducible submodule. Then Lemma says that in $K_0(A\text{-grmod})$ we have

classes of irreps in deg 0

$$[M] \in [L] + v^{-1} Span_{\mathbb{Z}[v^{-1}]}([L'])$$

This is (i).

Now we proceed to (ii). This requires the examination of duality. We note that if M is an A-module, M^* is an A^{opp}-module. If M is graded, $M = \bigoplus_{i \in \mathbb{Z}} M_i$, then M_i^* is graded: $(M^*)_i := (M_{-i})^*$ (so that the pairing map $M \otimes M^* \longrightarrow \mathbb{C}$ has degree 0). So we get a contravariant equivalence $A\text{-grmod} \longrightarrow A^{opp}\text{-grmod}$. If we have a graded algebra isomorphism, say $\varphi: A \xrightarrow{\sim} A^{opp}$, we get a contravariant self-equivalence of A-grmod ($M \mapsto M^*$ w. action twisted by φ) to be denoted by $\widetilde{\mathbb{D}}_\varphi$

Exercise: $[\tilde{\mathbb{D}}_\varphi]: K_0(A\text{-grmod}) \longrightarrow K_0(A\text{-grmod})$ is $\mathbb{Z}[v^{\pm 1}]$-semilinear (w.r.t. $v \mapsto v^{-1}$)

$\tilde{\mathbb{D}}_\varphi$ sends an irreducible in deg 0 to an irreducible in deg 0. Now suppose that $\tilde{\mathbb{D}}_\varphi$ fixes all irreps. Then the basis $[L]$ of $K_0(A\text{-grmod})$ is fixed by $[\tilde{\mathbb{D}}_\varphi]$.

Let's explain how the identification $A \longrightarrow A^{opp}$ works. We have $A = \mathrm{End}_R(\bigoplus_{w\in W} \underline{B}_w)^{opp}$, where $R = \mathbb{C}[\mathfrak{h}^*]^{coW}$. This realization gives an identification $A \xrightarrow{\sim} A^{opp}$ of graded algebras. First of all, note that $A^{opp} = \mathrm{End}_R(\bigoplus_{w\in W} \underline{B}_w^*)$, where $\underline{B}_w^*$ is dual graded module. Now to identify A w. A^{opp} it's enough to establish a graded R-module isomorphism $\underline{B}_w \simeq \underline{B}_w^*$. The corresponding equivalence $\tilde{\mathbb{D}}_\varphi$ is forced to fix each deg 0 irreducible A-module (exercise).

To show that $\underline{B}_w \simeq \underline{B}_w^*$ we can argue as follows.

Lemma: We have a graded R-module isomorphism $\underline{BS}_w \simeq \underline{BS}_w^*$

Proof: It's enough prove that there's a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\underline{BS}_w$ (this yields a vector space isomorphism $\underline{BS}_w \xrightarrow{\sim} \underline{BS}_w^*$) s.t. $(rb, b') = (b, rb')$ (this condition implies that the isomorphism $\underline{BS}_w \xrightarrow{\sim} \underline{BS}_w^*$ is R-linear), and the degree of $(\cdot,\cdot)$ is 0 (so that $\underline{BS}_w \xrightarrow{\sim} \underline{BS}_w^*$ is graded. To prove the existence of a form we can argue inductively: suppose $M \in R\text{-grmod}$ has a degree 0 R-invariant symmetric bilinear form $(\cdot,\cdot)_M$. We define a form $(\cdot,\cdot)$ on $R\otimes_{R^s} M$ as follows. Recall, Section 1.1 of Lec 25, that $R\otimes_{R^s} M \simeq 1_s \otimes M \oplus h_s \otimes M$. We define

$(\cdot,\cdot)$ on $R\otimes_{R^s}M$ by setting:

$$(1_s\otimes m, 1_s\otimes m') = (h_s\otimes m, h_s\otimes m') = 0$$

$$(1_s\otimes m, h_s\otimes m') = (h_s\otimes m, 1_s\otimes m') := (m, m')_M.$$

It's an exercise to check that $(\cdot,\cdot)$ has required properties. □

Exercise: Use $\underline{BS}_w \simeq \underline{BS}_w^*$ and Theorem in Sec 1.2 of Lec 26 (and the Krull-Scmidt theorem) to check that $\underline{B}_w \simeq \underline{B}_w^*$.

Conclusion:

So, we have checked the properties (i) and (ii). This shows that under the identification of $K_0(A\text{-grmod}) \xrightarrow{\sim} \mathcal{H}_v(W)$ where the graded A-module corresponding to $\nabla(w\cdot\lambda^-) \in \mathcal{O}^\chi$ is sent to H_w the classes of deg 0 irreducible modules are the KL basis in the normalization above. To prove this we need to know that A is positively graded and there's an isomorphism $\varphi: A \to A^{opp}$ of graded algebras s.t. the resulting contravariant self-equivalence $\widetilde{\mathbb{D}}_\varphi$ of A-mod fixes each irreducible.