Representations of $\mathfrak{sl}_2$ (and related Lie algebras), part 2

0) Introduction

1) Definition.

2) Properties.

0) This afterlife lecture is a continuation of Lecture 14.5 (note that $29 = 14.5 \cdot 2$). In that lecture we were dealing w. certain categories $\mathcal{C}$ (e.g. $\mathcal{C} = \bigoplus_{n \geq 0} \mathbb{F}S_n\text{-mod}$), see Sec 1 of Lecture 14.5. We have discussed that an action of $\mathfrak{sl}_2$ on $\mathcal{C}$ should include exact endofunctors $E, F: \mathcal{C} \to \mathcal{C}$ and a decomposition $\mathcal{C} = \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{C}_\lambda$ s.t. $[E], [F]$ define the structure of a weight representation of $\mathfrak{sl}_2$ (see Def'n in Sec 1.2 of Lec 14.5) on (the complexification of) $K_0(\mathcal{C})$.

However, this definition is not particularly useful as it provides very few tools to study the functors E & F. We need more structure. This structure is a bunch of functor morphisms.

Let's explain an ideological reason for that. In a sentence: what acts is one level categorical structure up compared to what it acts on. For example, on a vector space (categorical level 0) we consider an action of an associative algebra. An associative ring can be thought of as a category w. single object whose set of morphisms is additionally equipped w. an abelian group structure so that the composition map is bi-additive. So it's categorical level 1. If we want to act on a category, what acts should be of categorical level 2:

have functors and their morphisms.

We follow J. Chuang, R. Rouquier "Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification" Ann. Math. 167 (2008).

1) Definition

1.1) Revisiting the example of $\bigoplus_n \mathbb{F}S_n$-mod

Let $\mathbb{F}$ be an algebraically closed field and α be an element in $\mathbb{Z}1 \subset \mathbb{F}$. Let $\mathcal{C} = \bigoplus_{n\geqslant 0} \mathbb{F}S_n\text{-mod}$. In Sec 1.3 of Lec 14.5 we have considered endofunctors $E_\alpha = \bigoplus_n (\mathrm{Res}^n_{n-1}\bullet)_\alpha$, $F_\alpha = \bigoplus_n (\mathrm{Ind}^n_{n-1}\bullet)_\alpha$ of $\mathcal{C}$. By the construction, compare to Lemma 6.9 in [RT1], F_α is left adjoint to E_α. Also, by the construction, E_α comes with a distinguished functor endomorphism: for $M \in \mathbb{F}S_n\text{-mod}$ ($\subseteq_\oplus \mathcal{C}$), $\mathcal{J}_n$ acts on $E_\alpha M$. The action of $\mathcal{J}_n - \alpha$ on $E_\alpha M$ is nilpotent.

We will need two more properties.

Lemma 1: In the above notation, $E_\alpha^2 M \subset M$ is $(n-1,n)$-stable

Proof: The degenerate affine Hecke algebra $\mathcal{H}(2)^{opp}$ acts on $\mathrm{Res}^{n-2}_n(M)$ w. $X_1 \mapsto \mathcal{J}_n$, $X_2 \mapsto \mathcal{J}_{n-1}$, $T \mapsto (n-1,n)$. The symmetric polynomials in X_1, X_2 are central. It follows that any simultaneous eigenspace for these polynomials is stable under any element of $\mathcal{H}(2)$ including T. To finish the proof note that $E_\alpha^2 M$ is such a generalized eigenspace (it corresponds to evaluating the symmetric polynomials in X_1, X_2 at the point (α,α)). $\square$

Lemma 2: F_α is isomorphic to the right adjoint of E_α.

Proof: Note that $\bigoplus_{\alpha\in\mathbb{F}} F_\alpha = \mathrm{Ind}_{n-1}^n$ is right adjoint to $\bigoplus_{\alpha\in\mathbb{F}} E_\alpha$. To show that F_α is isomorphic to the right adjoint of E_α it's enough to show that

(*) the isomorphism $\mathrm{Hom}_{S_{n-1}}(EM, N) \xrightarrow{\sim} \mathrm{Hom}_{S_n}(M, FN)$ restricts to $\mathrm{Hom}_{S_{n-1}}(E_\alpha M, N) \xrightarrow{\sim} \mathrm{Hom}_{S_n}(M, F_\alpha N)$.

For this we decompose $\mathbb{F}S_n$-mod w.r.t. the generalized eigenspaces for the action of the central subalgebra of symmetric polynomials in the elements $\mathcal{J}_1, \dots, \mathcal{J}_n$: $\mathbb{F}S_n\text{-mod} = \bigoplus_{\underline{\alpha}} \mathbb{F}S_n\text{-mod}_{\underline{\alpha}}$, where the summation is taken over unordered n-tuples of elements of $\mathbb{Z}\cdot 1$ and $\mathbb{F}S_n\text{-mod}_{\underline{\alpha}}$ is the full subcategory of all $\mathbb{F}S_n$-modules, where $f(\mathcal{J}_1, \dots, \mathcal{J}_n)$ acts w. single eigenvalue $f(\underline{\alpha})$, $\forall$ symmetric polynomials f.

Exercise: • Show E_α sends $\mathbb{F}S_n\text{-mod}_{\underline{\alpha}}$ to $\mathbb{F}S_{n-1}\text{-mod}_{\underline{\alpha}'}$, where $\underline{\alpha}'$ is obtained from $\underline{\alpha}$ by removing one copy of α (if $\underline{\alpha}$ contains such, otherwise E_α sends $\mathbb{F}S_n\text{-mod}_{\underline{\alpha}}$ to 0).

• Use that F_α is left adjoint to E_α to show that F_α sends $\mathbb{F}S_{n-1}\text{-mod}_{\underline{\alpha}'}$ to $\mathbb{F}S_n\text{-mod}_{\underline{\alpha}''}$, where $\underline{\alpha}''$ is obtained from $\underline{\alpha}'$ by adding a copy of α.

• Deduce (*) and finish the proof $\square$

1.2) Definition of an $\mathfrak{sl}_2$-action on $\mathcal{C}$

Let $\mathcal{C}$ be as above: the direct sum of categories of modules over finite dimensional algebras.

Definition: An action of $\mathfrak{sl}_2$ on $\mathcal{C}$ consists of:

Data: (i) A pair of exact endofunctors E, F of $\mathcal{C}$

(ii) A direct sum decomposition $\mathcal{C} = \bigoplus_{m \in \mathbb{Z}} \mathcal{C}_m$,

(iii) Endomorphisms $X \in End(E)$, $T \in End(E^2)$,

(iv) A fixed isomorphism between F and the left adjoint of E, i.e. adjunction unit $\varepsilon : id \Rightarrow FE$ and counit $\eta : EF \Rightarrow id$,

that satisfies the following

Axioms: (I) F is isomorphic to the right adjoint of E.

(II) Consider endomorphisms $X1, 1X, T \in End(E^2)$ (where $X1$ acts by X on the first application of E, and $1X$ acts on the second). They satisfy the relations of $\mathcal{H}(2)^{opp}$:

$$T(X1) = (1X)T + 1.$$

(III) $\exists\, \alpha \in \mathbb{F}$ s.t $X_M - \alpha \in End(EM)$ is nilpotent, $\forall M \in \mathcal{C}$.

(IV) The operators $[E], [F]$ & the direct sum decomposition $K_0(\mathcal{C}) = \bigoplus_m K_0(\mathcal{C}_m)$ give the structure of a weight representation of $\mathfrak{sl}_2$. Moreover, this representation is integrable meaning that $\forall\, v \in K_0(\mathcal{C})\ \exists\, d (= d_v) \in \mathbb{Z}_{>0}$ s.t. $[F]^d v = [E]^d v = 0$.

Exercise: Show that the assignment $X_i \mapsto \underbrace{1..1}_{i-1} X \underbrace{1...1}_{d-i}$, $i = 1, \dots, d$, $T_j \mapsto \underbrace{1..1}_{j-1} T \underbrace{1...1}_{d-j-1}$ defines an algebra homomorphism

$$\mathcal{H}(d)^{opp} \longrightarrow End(E^d)$$

for all $d > 0$ (compare to the solution to Prob 5 in HW3).

Example: for each $\alpha\in\mathbb{Z}\cdot 1\subset\mathbb{F}$ we have an action of $\mathfrak{sl}_2$ on $\mathcal{C}=\bigoplus_{n\geqslant 0}\mathbb{F}S_n\text{-mod}$ with the functors being E_α, F_α. The direct sum decomposition $\mathcal{C}=\bigoplus_m \mathcal{C}_m$ was introduced in Section 1.3.1 of Lec 14.5. Element $X\in End(E)$ is as follows. On $M\in\mathbb{F}S_n\text{-mod}$, it's defined by X_M = action of $\mathcal{J}_n$ of M. Similarly, $T\in End(E_\alpha^2)$ comes from the action of $(n-1,n)$. The isomorphism between F_α and the left adjoint of E_α was discussed in the beginning of Sec 1.1 of this lecture.

Let's explain why the axioms hold. For (IV), the claim about the weight representation was checked in Lecture 14.5, Section 1.3. In the construction we have established an $\mathfrak{sl}_2$-linear epimorphism $\mathcal{F}\twoheadrightarrow\bigoplus_{n\geqslant 0}K_0(\mathbb{F}S_n\text{-mod})$. It's an exercise to check that the $\mathfrak{sl}_2$-action on $\mathcal{F}$ is integrable.

(I) is Lemma 2 in Sec. 1.1. (II) follows from relations between $\mathcal{J}_{n-1}, \mathcal{J}_n, (n,n-1)$ established in Lemma 4.1, [RT1]. (III) follows from the construction of X.

1.3) Actions of other Lie algebras.

Let $\mathfrak{g}=\hat{\mathfrak{sl}}_p$ if $\operatorname{char}\mathbb{F}=p$ & $\mathfrak{g}=\mathfrak{sl}_\infty$ if $\operatorname{char}\mathbb{F}=0$. The former algebra was discussed in Section 3.1 of Lecture 20. The latter algebra is an infinite version of $\mathfrak{sl}_n$, it consists of all matrices $(m_{ij})_{i,j\in\mathbb{Z}}$ s.t. only finitely many m_{ij}'s are nonzero & $\sum_{i\in\mathbb{Z}} m_{ii}=0$. It's a Kac-Moody algebra corresponding to the Dynkin diagram, whose nodes are $\mathbb{Z}$ and we have an edge between i & j iff $|i-j|=1$. Note that in both cases the nodes of the Dynkin diagram are in a natural bijection w. $\mathbb{Z}\cdot 1\subset\mathbb{F}$.

Let's explain a definition of a $\mathfrak{g}$-action on $\mathcal{C}$. It closely follows the definition in Sec 1.2. We need to replace (ii) with the decomposition $\mathcal{C}=\bigoplus_{\lambda\in\Lambda}\mathcal{C}_\lambda$, where Λ is the weight lattice of $\mathfrak{g}$.

Axioms (I) and (II) are as before. Axiom (III) now says that all eigenvalues of X_M are in $\mathbb{Z}\cdot 1(\subset\mathbb{F})$. Then we can decompose EM into the direct sum of eigen-spaces $(EM)_\alpha$ for X_M. The assignment $M\mapsto(EM)_\alpha$ is a functor (exercise) so we get a functor $E_\alpha:\mathcal{C}\to\mathcal{C}$. All these functors are exact and $E=\bigoplus_{\alpha\in\mathbb{Z}1}E_\alpha$.

By the adjunction in (I), we have an identification $\operatorname{End}(F)\xrightarrow{\sim}\operatorname{End}(E)^{opp}$. So we can consider X as an endomorphism of F & use it to decompose F as $\bigoplus_\alpha F_\alpha$.

Exercise: F_α is left adjoint to E_α.

Next, axiom (IV) needs to be replaced with the condition that $[E_\alpha],[F_\alpha]$ & $K_0(\mathcal{C})=\bigoplus_{\lambda\in\Lambda}K_0(\mathcal{C}_\lambda)$ equip $K_0(\mathcal{C})$ with the structure of a weight $\mathfrak{g}$-representation. Moreover, we require that the representation is integrable: $\forall\alpha$ & $v\in K_0(\mathcal{C})$ $\exists\, m>0$ s.t. $[E_\alpha]^m v=$ $=[F_\alpha]^m v=0$.

Exercise:
- F_α is isomorphic to the right adjoint of E_α;
- T preserves the direct summand $E_\alpha^2\subseteq_\oplus E^2$.
- The following data define a categorical action of $\mathfrak{sl}_2$

on $\mathcal{C}$: E_α, F_α, the images of X in $End(E_\alpha)$, T in $End(E_\alpha^2)$, the isomorphism between F_α and the left adjoint of E_α and finally the decomposition $\mathcal{C} = \bigoplus_{m\in\mathbb{Z}} \mathcal{C}_m$, where $\mathcal{C}_m$ is the direct sum of all $\mathcal{C}_\lambda$ s.t. the pairing of λ w. the simple root corresponding to α is m.

Example 1: $\mathcal{C} = \bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod is a categorical $\mathfrak{g}$-representation w. $E = \bigoplus_{n\geqslant 0} Res^n_{n-1}$, $F = \bigoplus_{n\geqslant 0} Ind^n_{n-1}$. The other pieces of data are as in Example in Sec 1.2.

It turns out that $K_0(\mathcal{C})$ is the irreducible highest weight representation of highest weight ω_0. The highest weight space is $K_0(\mathbb{F}S_0\text{-mod})$.

Example 2: Consider the category $\mathcal{O}$ for $\mathfrak{gl}_n$ (the weights are now in $\mathbb{Z}^n \subset \mathfrak{h}^*$, where $\mathfrak{h} \subset \mathfrak{gl}_n$ is still the subalgebra of diagonal matrices). Define endofunctors $E = V\otimes\bullet$, $F = V^*\otimes\bullet$ of $\mathcal{O}$ and endomorphisms $X = End(E)$, the tensor Casimir from Prob 5 in HW3, and $T\in End(E^2)$ permutes the tensor factors V. This is a part of a categorical $\mathfrak{sl}_\infty$-action. The weight decomposition of $\mathcal{O}$ is as follows. Let $\lambda = \sum_{i\in\mathbb{Z}} a_i\omega_i$. The $\mathcal{O}_\lambda$ is the infinitesimal block of the Verma module w. highest weight $\mu_\lambda - \tilde{\rho}$, where:

- μ_λ is an n-tuple, where the entry i occurs a_i times
- $\tilde{\rho} = (n-1, n-2, \dots, 1, 0)$.

We then identify $K_0(\mathcal{O})$ w. $(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$, where $\mathbb{C}^{\mathbb{Z}}$ is a tautological representation of $\mathfrak{sl}_\infty$. Under this identification $[\Delta(\mu-\rho)]$ corresponds to the tensor monomial $e_{\mu_1}\otimes\ldots\otimes e_{\mu_n}$, where $\mu=(\mu_1,\ldots,\mu_n)$ and we write $e_j, j\in\mathbb{Z}$, for the tautological basis element of $\mathbb{C}^{\mathbb{Z}}$. To check the axioms is a meaningful and not quite trivial exercise (based on problems 2 and 5 of HW3).

Remarks: 1) In the definition of a categorical $\mathfrak{g}$-representation one can replace the degenerate affine Hecke algebra relations w. the genuine affine Hecke algebra relations (for some $q=\varepsilon\in\mathbb{F}^\times$ in axiom II.

An example of a categorical $\hat{\mathfrak{sl}}_\ell$-action is on $\bigoplus_{n\geqslant 0} \mathcal{H}_\varepsilon(S_n)$-mod, where ε is a primitive ℓth root of 1.

2) One can define a notion of a categorical $\mathfrak{g}$-representation for an arbitrary Kac-Moody algebra $\mathfrak{g}$. But one needs to replace the (degenerate) affine Hecke algebras (that are very classical) with the so called KLR (Khovanov-Lauda-Rouquier) algebras constructed in [R]: R. Rouquier "2-Kac-Moody algebras" arXiv: 0812.5023 & [KL]: M. Khovanov, A. Lauda "A diagrammatic approach to categorification of quantum groups" Trans. Amer. Math. Soc. 363 (2011) specifically for the categorification purpose.

2) Properties

Here we explain some easy to state properties of a categorical $\mathfrak{sl}_2$-representations in $\mathcal{C}$. For more, including a construction of derived self-equivalence of $\mathcal{C}$ from the $\mathfrak{sl}_2$-action see the Chuang-Rouquier paper, [CR]. The proofs crucially use the representation theory of (degenerate) affine Hecke algebras.

2.1) Simple subs & quotients.

The following is Proposition 5.20 in [CR]. It implies Theorem 6.15 in [RT1].

Theorem: Let $\mathcal{C}$ be equipped w. an action of $\mathfrak{sl}_2$ according to the definition in Sec 1.2. Let $L \in Irr(\mathcal{C})$. Then EL has the unique irreducible subobject and the unique irreducible quotient, and they are isomorphic.

The same holds for FL - in fact, these two functors are interchangeble.

2.2) Categorification of $[e,f]=h$.

We know that $[E][F]-[F][E]=\lambda$ on $K_0(\mathcal{C}_\lambda)$. This lifts to a level of categories as follows, [CR], Section 5.5.

Theorem: We have the following isomorphisms of endo-functors of $\mathcal{C}_\lambda$:

$$EF \simeq FE \oplus id^{\oplus \lambda} \quad \text{if } \lambda \geqslant 0$$

$$EF \oplus id^{\oplus -\lambda} \simeq FE \quad \text{if } \lambda \leqslant 0$$

2.3) Minimal categorifications.

By a minimal $\mathfrak{sl}_2$-categorification we mean a category $\mathcal{C}$ w. an action of $\mathfrak{sl}_2$ s.t.

- $K_0(\mathcal{C})$ is a finite dimensional irreducible $\mathfrak{sl}_2$-representation.
- For its highest weight λ, $\mathcal{C}_\lambda \simeq \mathrm{Vect}$.

The following is a consequence of [CR], Proposition 5.26. It is an analog of the classification of finite dimensional irreducible $\mathfrak{sl}_2$-irreps.

Theorem: Minimal categorifications are classified by their highest weights.

One can construct the minimal categorification, $\mathcal{C}(\lambda)$, w. highest weight λ in several equivalent ways. For example,

$$\mathcal{C}(\lambda) = \bigoplus_{k=0}^{\lambda} H^*(Gr(k,\lambda))\text{-mod},$$

where $Gr(k,\lambda)$ is the Grassmanian of k-dimensional subspaces in $\mathbb{C}^\lambda$.

A direct analog of Theorem for general Kac-Moody algebras was established by Rouquier in [R], Section 5.1.2. For example, for type A Lie algebras $\mathfrak{g}$ the minimal categorifications arise from the representation categories of "cyclotomic" (degenerate) Hecke algebras: $\bigoplus_{n\geq 0} \mathbb{F}S_n\text{-mod}$ is an example.