(Categorical) Hecke algebras & link invariants.

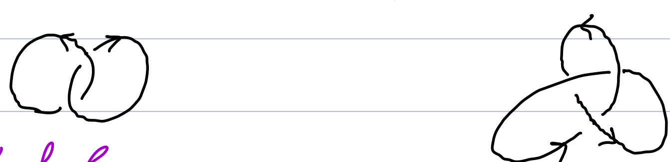
1) Knots, links & braids.

2) HOMFLY polynomial.

1) We start by introducing some terminology. A **link** (w. k components) is a continuous embedding of $S^1 \sqcup \ldots \sqcup S^1$ (k components) into $\mathbb{R}^3$. Links are considered up to isotopy (a continuous family of diffeomorphisms) of $\mathbb{R}^3$. A link w. one component is called a **knot**.

We will consider oriented links.

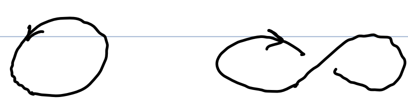
Usually, links are presented by their **diagrams**: images under sufficiently general projections $\mathbb{R}^3 \to \mathbb{R}^2$. For example:



(1)

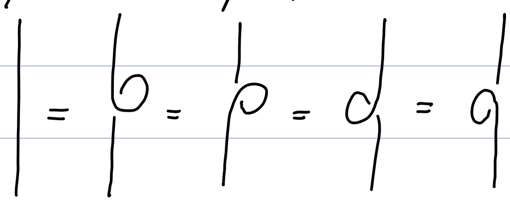
Hopf link　　　　Trefoil

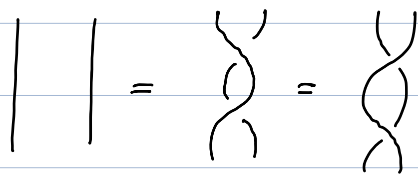
One can similarly talk about isotopies of diagrams and isotopic diagrams give isotopic links. But the converse is not true, for example both diagrams below come from the "unknot" (the trivial embedding $S^1 \hookrightarrow \mathbb{R}^3$):

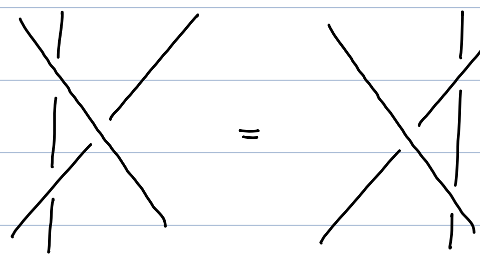


(2)

It's known when two diagrams give rise to isotopic links: this is true if and only if the diagrams are obtained from one another by a sequence of so called **Reidemeister moves** depicted below (and planar isotopies):

R1: 

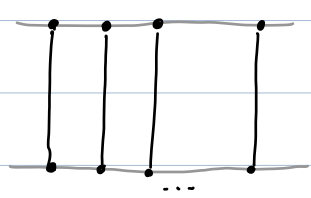
R2: 

R3:  (& variants of this, compare to R1)

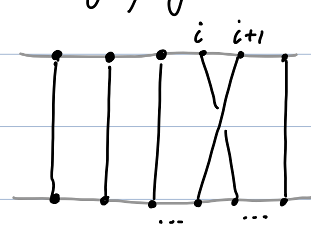
One can put orientations in R1-R3 arbitrarily.

For example, the two diagrams in (2) are obtained from one another using (R2).

Now we pass to braids. A braid is a configuration of strands in $\mathbb{R}^2 \times [0,1]$ connecting n fixed points in $\mathbb{R}^2 \times \{0\}$ to n fixed points in $\mathbb{R}^2 \times \{1\}$ so that the strands do not intersect and each projects isomorphically to $[0,1]$. Braids are viewed up to isotopy. They also can be presented by 2D diagrams (by projection to $\mathbb{R} \times [0,1]$), e.g.



trivial braid



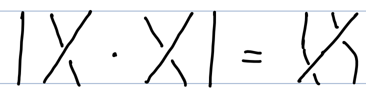
the braid T_i, $i=1,..n-1$

(3)

We orient braids bottom to top.
It turns out that

Proposition: two braids are equal iff their diagrams are obtained from one another by a sequence of planar isotopies and the analogs of R2 & R3.

Braids form a monoid. The composition is the vertical stacking, e.g.

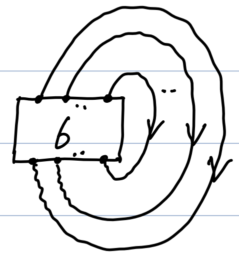


and the unit is the trivial braid. This monoid is generated by the elements T_i, $i=1,\dots,n-1$, and T_i^{-1} which are different from T_i by the order of crossing. The notation is suggestive: we have $T_i T_i^{-1} = T_i^{-1} T_i = 1$ thx to R2. So the braids form a group.
The proposition above can be restated as follows:

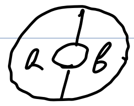
Proposition': the braid group, Br_n, on n strands is generated by T_i's w. relations:

- $T_i T_j = T_j T_i$, $|i-j|>1$ (this is a planar isotopy)
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ (this is R3)

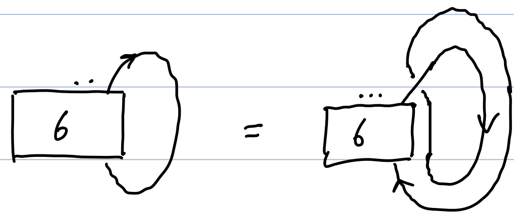
Finally, let's discuss a connection between braids & links. For a braid b we can define its closure $\overline{b}$ as follows:

$\bar{\sigma} =$ 

Different braids can give the same link. For example, let $a, b \in Br_n$. Then $\overline{ab} = \overline{ba}$. To see this note that both sides can be depicted as



Also, let note that $Br_n \hookrightarrow Br_{n+1}$ so we can view any $\sigma \in Br_n$ as an element of Br_{n+1}. We claim that $\overline{\sigma T_n^{\pm 1}} = \bar{\sigma}$. E.g. for the + sgn, have



Definition: By Markov's moves one means the following operations:

(M1) for $a, b \in Br_n$ replace ab w. ba.

(M2) for $\sigma \in Br_n$ replace σ w. $\sigma T_n^{\pm 1} \in Br_{n+1}$ or vice versa.

Facts: I: Alexander – every link is the closure of some braid.

II: Markov – if two braids give the same link, then one is obtained from the other by a sequence of Markov moves.

2) HOMFLY polynomial.

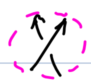
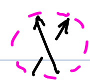
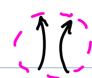
There's no algorithmic way to decide when two diagrams represent isotopic links. We can try to address a weaker question: from a diagram produce a computable quantity that is the same when two diagrams represent isotopic links. A famous example of such an invariant is HOMFLY polynomial.

Theorem/definition: There is the unique map

$$L \mapsto P(L) : \{\text{Links}\} \longrightarrow \mathbb{Z}[a^{\pm 1}, q^{\pm 1}][(q-q^{-1})^{-1}]$$

(where a, q are independent variables) satisfying the following two conditions:

- $P(\text{unlink w. } k \text{ components}) = \left(\frac{a-a^{-1}}{q-q^{-1}}\right)^k$
- Suppose L_+, L_-, L_0 are three links whose diagrams are the same outside of a small circle, and inside of the circle we have

L_+  L_-  L_0 

Then we have the relation $a^{-1}P(L_+) - aP(L_-) = (q^{-1}-q)P(L_0)$

We say that $P(L)$ is the HOMFLY polynomial of L

Example: Let's compute the HOMFLY polynomial of the Hopf link:

L_- $\rightsquigarrow$ L_+: unlink w 2 components L_0: unknot

$$P(L_-) = a^{-2}P(L_+) + a^{-1}(q-q^{-1})P(L_0) = a^{-2}\left(\frac{a-a^{-1}}{q-q^{-1}}\right)^2 + a^{-1}(a-a^{-1}).$$

Note that if P exists, then it's unique: L_0 has one less crossing than L_+ & L_-, and we can unlink every link by exchanging L_+ w. L_- and vice versa.

The existence of P is nontrivial. It can be constructed using the R-matrix for the quantum group $U_q(\mathfrak{sl}_n)$, see Sec 15.2 in V. Chari, A. Pressley. "A guide to quantum groups."

Now we explain a connection to Hecke algebras. By Fact II in the previous section, we can view P as a map

$$\bigsqcup_n Br_n \to \mathbb{Z}[a^{\pm 1}, q^{\pm 1}][(q-q^{-1})^{-1}]$$

Extend P by additivity to $\mathbb{Z}Br_n$. for all $\alpha, \beta \in Br_n$ we have

$a^{-1}P(\alpha T_i \beta) - a\,P(\alpha T_i^{-1}\beta) = (q^{-1}-q)\,P(\alpha\beta) \Leftrightarrow$

$P(\alpha(a^{-1}T_i - aT_i^{-1} - (q^{-1}-q))\beta) = 0$

So $P: \mathbb{Z}Br_n \longrightarrow \mathbb{Z}[a^{\pm 1}, q^{\pm 1}][(q-q^{-1})^{-1}]$ factors through the quotient of $\mathbb{Z}Br_n$ by the relation $a^{-1}T_i - aT_i^{-1} = (q-q^{-1})$, $i=1,\dots n$. This quotient is nothing else but the Hecke algebra $\mathcal{H}_q(S_n)$.

Rem*: "Categorification" allows to streghthen the HOMFLY polynomial to a function of three variables: a, q, z, known as the Khovanov-Rozansky homology. Even better, the KR homology is the homology of a complex of bigraded vector spaces and the invariant in question encodes the dimensions of bigraded pieces of the individual cohomology groups. Up to a normalization, P is obtained as the Euler characteristic of the complex (reducing the number of variables from 3 to 2). There's a construction of the KR homology due to Khovanov: M. Khovanov, Triply graded link homology and Hochschild homology of Soergel bimodules, arXiv: 0510265 based on a construction of Rouquier, R. Rouquier, Categorification of the braid groups, arXiv: 0409593. Namely, Rouquier defined a "group homomorphism" $Br_n \to K^b(SBim)$, where the target is the bounded homotopy category of the category SBim of Soergel bimodules. Khovanov then showed that applying the Hochshild cohomology functor to the Rouquier complexes leads to a link invariant that coincides with the one previously defined by Khovanov & Rozansky.