Part II: representation of $U_q(\mathfrak{g})$

Assume $q \in k^\times$ is <u>not</u> a root of unity and $\mathrm{char}(k) \neq 2$

For any U-module M, and any $\lambda \in \Lambda$ and $\sigma : \mathbb{Z}\Phi \longrightarrow \{\pm 1\}$ group homomorphism, let

$$M_{\lambda,\sigma} := \{ m \in M \mid K_\gamma m = \sigma(\gamma) q^{(\lambda,\gamma)} m \}$$

<u>Proposition</u> Let M be a f.d. U-module, then

$$M = \bigoplus_{\lambda,\sigma} M_{\lambda,\sigma}$$

and $E_\alpha(M_{\lambda,\sigma}) \subset M_{\lambda+\alpha,\sigma}$, $F_\alpha(M_{\lambda,\sigma}) \subset M_{\lambda-\alpha,\sigma}$

Pf Exercise (use the case $\mathfrak{g} = \mathfrak{sl}_2$)

<u>Rmk</u> Here is something different from the classical case: the eigen-spaces for the Cartan part has to take into account this sign functions and not just weights.

Let $M^\sigma := \bigoplus_\lambda M_{\lambda,\sigma}$ $\quad$ (so $M = \bigoplus_\sigma M^\sigma$)

We say that M is <u>of type σ</u>,
if $M = M^\sigma$, and set $U\text{-mod}^\sigma$ the
full subcategory of U-mod of type σ modules

<u>Rmk 1)</u> Let $\sigma \neq \tau$, and $M = M^\sigma$, $N = N^\tau$
then $\mathrm{Hom}_U(M, N) = 0$

So $U\text{-mod} = \bigoplus_\sigma U\text{-mod}^\sigma$

<u>Rmk 2)</u> Given any $M \in U$-mod, we can
twist the U-action by $\tilde{\sigma} : U \to U$
sending $E_\alpha \mapsto \sigma(\alpha) E_\alpha$, $F_\alpha \mapsto F_\alpha$, $K_\alpha \mapsto \sigma(\alpha) K_\alpha$
Twisting by $\tilde{\sigma}$ gives isomorphisms of categories

$$U\text{-mod}^1 \rightleftarrows U\text{-mod}^\sigma$$

Hence we can restrict ourselves to $U\text{-mod}^1$.
(and denote $M_\lambda := M_{\lambda,1}$)

Tensor products

Since U is a Hopf algebra, for any $M, N \in U\text{-mod}$, we can give $M \otimes N$ the structure of a U-module

(it is naturally a $U \otimes U$-module, and we use Δ, as in the adjoint representation)

So if $\Delta(u) = \sum_i u_i \otimes u_i'$, we have

$$u \cdot (m \otimes n) := \sum u_i m \otimes u_i' n$$

Rmk 1 $\quad M_{\lambda,\sigma} \otimes N_{\mu,\tau} \subset (M \otimes N)_{\lambda+\mu, \sigma\tau}$

Hence, in particular, $\otimes$-product of type 1 modules is of type 1, i.e. $U\text{-mod}^1$ is a monoidal subcategory of U-mod.

Rmk 2 (important!) The natural map

$$\begin{aligned} M \otimes N &\longrightarrow N \otimes M \\ m \otimes n &\longmapsto n \otimes m \end{aligned}$$

is not a morphism of U-modules because Δ is not cocommutative

(recall for example the formulas for the coproduct $\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha \ldots$)

Trivial module and dual modules

The counit $\varepsilon : U \longrightarrow k$ gives k the structure of U-module and we call it <u>trivial module</u> $(\forall x \in k,\ u \cdot x = \varepsilon(u) x)$

The antipode $S : U \longrightarrow U^{op}$ allows to give to $M^* = \mathrm{Hom}_k(M, k)$ (for any $M \in U\text{-mod}$) the structure of a U-module, namely

$$(u \cdot f)(m) := f(S(u) m)$$

One has $(M^*)_{\lambda,\sigma} = \{ f \in M^* \mid f(M_{\lambda',\sigma'}) = 0 \ \forall (\lambda',\sigma') \neq (-\lambda,\sigma) \}$

$\Rightarrow$ $U\text{-mod}^1$ is stable under $(-)^*$

<u>Rmk</u> The usual k-linear morphism (iso if $\dim M < \infty$)

$$\begin{aligned} M &\longrightarrow (M^*)^* \\ m &\longmapsto (f \mapsto f(m)) \end{aligned} \quad \text{is } \underline{\text{not}}\ U\text{-linear}$$

but using the description of S^2 (from Part I) we get one by

$$\begin{aligned} M &\longrightarrow (M^*)^* \\ m &\longmapsto (f \mapsto f(K_{2\rho}^{-1} m)) \end{aligned}$$

Hopf algebra axioms $\Rightarrow$

$$\begin{aligned} M^* \otimes M &\longrightarrow k \\ f \otimes m &\longmapsto f(m) \end{aligned}$$

morphism of U-modules

but to switch M and M^*, use the <u>Rmk</u> and set

$$\begin{aligned} M \otimes M^* &\longrightarrow k \\ m \otimes f &\longmapsto f(K_{2\rho}^{-1} m) \end{aligned}$$

Hom spaces

We also give $\mathrm{Hom}_k(M,N)$ (for $M,N \in U\text{-mod}$) the structure of U-module, first considering the structure of a $(U\otimes U)$-module given by

$$((u_1\otimes u_2)\cdot \varphi)(m) = u_1\,\varphi(S(u_2)m)$$

and then using Δ

In this way
$$N\otimes M^* \longrightarrow \mathrm{Hom}_k(M,N)$$
$$n\otimes f \longmapsto (m \mapsto f(m)n)$$
is a morphism of U-modules (iso if $\dim M$, $\dim N < \infty$)

Denote, for $M \in U\text{-mod}$,
$$M^U := \{m \in M \mid um = \varepsilon(u)m\}$$
Then one can check $\mathrm{Hom}_k(M,N)^U = \mathrm{Hom}_U(M,N)$

<u>Rmk</u> $\mathrm{Hom}_k(M,\mathrm{Hom}_k(N,V)) \xrightarrow{\sim} \mathrm{Hom}_k(M\otimes N, V)$ is an isomorphism of U-modules, so it preserves U-invariants
$$\Rightarrow \mathrm{Hom}_U(M,\mathrm{Hom}_k(N,V)) \xrightarrow{\sim} \mathrm{Hom}_U(M\otimes N, V)$$

Quantum trace

Usual $\mathrm{tr} : \mathrm{End}_k(M) \longrightarrow k$ ($\dim M < \infty$) is not U-linear but we can take $\mathrm{tr}_q : \varphi \longmapsto \mathrm{tr}(\varphi \circ K_{2\rho}^{-1})$
$(\mathrm{End}_k(M) \simeq M\otimes M^* \dots)$

Classification of simple finite dimensional U-modules

Lemma M f.d. U-module $\neq 0$:

a) $\exists \lambda \in \Lambda, v \in M_\lambda$ s.t. $E_\alpha v_\lambda = 0 \quad \forall \alpha \in \Pi$

b) If λ and v_λ are as in (a) then $\lambda \in \Lambda^+$ and
$F_\alpha^{\langle \lambda, \alpha^\vee \rangle + 1} v_\lambda = 0$

Pf Exercise (do the $\mathfrak{sl}_2$ case first)

Verma modules Set $M(\lambda) := U \Big/ \left(\sum_{\alpha\in\Pi} U E_\alpha + \sum_{\alpha\in\Pi} U(K_\alpha - q^{(\alpha,\lambda)}) \right)$

$v_\lambda :=$ class of $1_U \Rightarrow v_\lambda \in M(\lambda)_\lambda$, $E_\alpha v_\lambda = 0$
$\forall \alpha \in \Pi$

we have
$$\begin{aligned} U^- &\xrightarrow{\sim} M(\lambda) \\ u &\longmapsto u v_\lambda \end{aligned}$$
isomorphism of U^--modules

Properties

1) (Universal property) $M \in U$-mod, $v \in M_\lambda$
w/ $E_\alpha v = 0 \quad \forall \alpha \in \Pi$
$\Rightarrow \exists!\ M(\lambda) \longrightarrow M$ sending v_λ to v.

2) $M(\lambda)$ has a unique simple quotient
$L(\lambda)$, and
$\mathrm{End}_U(M(\lambda)) = \mathrm{End}_U(L(\lambda)) = k.$

Several results of the classical case (category $\mathcal{O}$) apply to this setting

<u>Proposition</u> $\lambda \in \Lambda^+$ (dominant wt).

Let $\varphi_\alpha : M(\lambda - (\langle \lambda, \alpha^\vee \rangle + 1)\alpha) \longrightarrow M(\lambda)$

$$v_{\lambda - (\langle \lambda, \alpha^\vee \rangle + 1)\alpha} \longmapsto F_\alpha^{\langle \lambda, \alpha^\vee \rangle + 1} v_\lambda$$

(check that it exists: this is just as in the classical case)

Then $\tilde{L}(\lambda) := M(\lambda) \Big/ \sum_{\alpha \in \Pi} \operatorname{im} \varphi_\alpha$

is finite dimensional.

<u>Rmk</u> ① There is a natural epimorphism $\tilde{L}(\lambda) \twoheadrightarrow L(\lambda)$

② Just as in the classical case, for $\tilde{L}(\lambda)$ the Weyl's character formula holds

<u>Rmk</u> Let $M \in U\text{-mod}$ be finite dimensional

Then $\dim M_\mu = \dim M_{w(\mu)}$

for any $\mu \in \Lambda$, $w \in W$

<u>Theorem</u> $\Lambda^+ \overset{1:1}{\longleftrightarrow} \{\text{simple f.d. } U\text{-modules of type 1}\} / \cong$

$$\lambda \longmapsto L(\lambda).$$

(again, this is just as in the classical case)

Examples

1) "Natural" representation for $U_q(\mathfrak{sl}_n)$

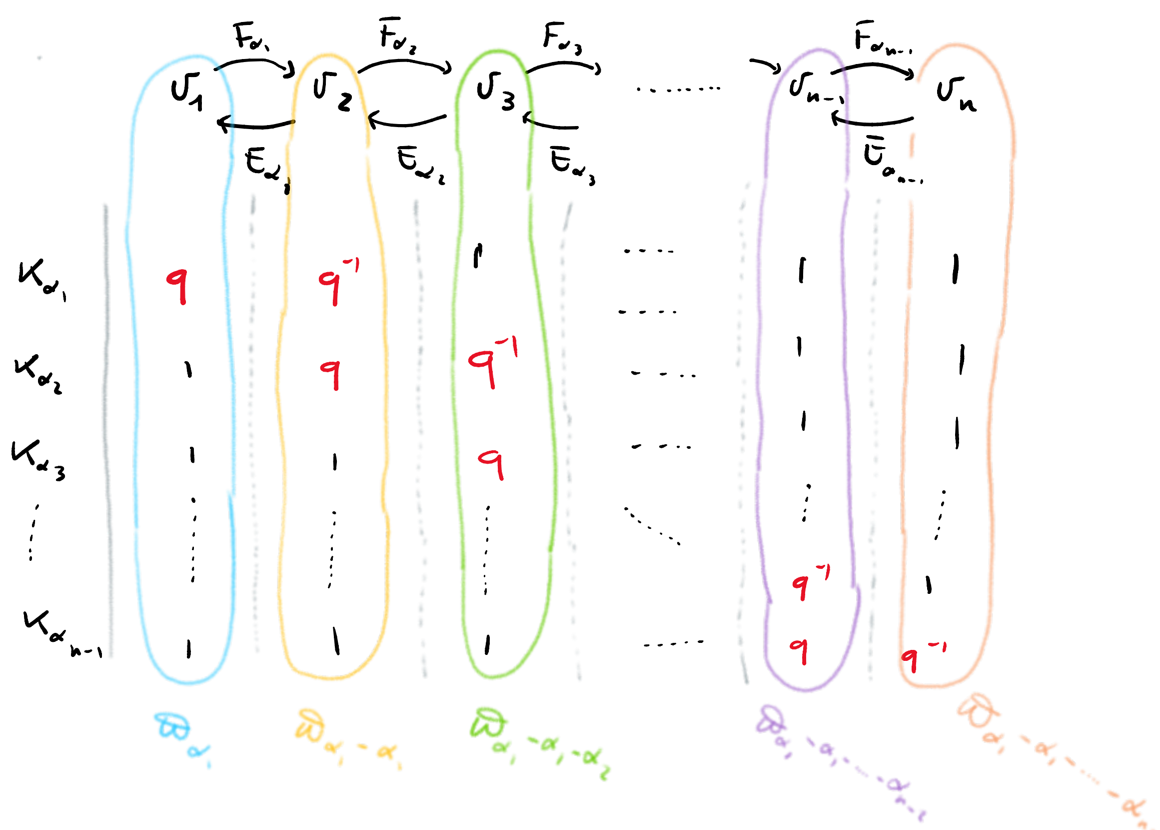
Consider a k-v.s. V of dim n w/ basis $v_1, \dots, v_n$ where $U_q(\mathfrak{sl}_n)$ acts as follows

$$K_{\alpha_i} \cdot v_j = \begin{cases} q v_j & \text{if } i=j \\ q^{-1} v_j & \text{if } i+1=j \\ v_j & \text{o/w} \end{cases} \qquad (i=1,\dots,n-1\,;\ j=1,\dots,n)$$

$$E_{\alpha_i} v_j = \begin{cases} v_{j-1} & \text{if } i=j-1 \\ 0 & \text{o/w} \end{cases}$$

$$F_{\alpha_i} v_j = \begin{cases} v_{j+1} & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$$

(one can check that this defines a $U_q(\mathfrak{sl}_n)$ representation that correspond to the natural representation of $\mathfrak{sl}_n$ in the classical case)



2) "Adjoint" representation of type A_2 $\quad \Pi = \{\alpha_1, \alpha_2\}$

$$V = \bigoplus_{\gamma \in \Phi} k v_\gamma \oplus k h_{\alpha_1} \oplus k h_{\alpha_2}$$

where $U_q(\mathfrak{sl}_3)$ acts as follows

$$K_{\alpha_i} v_\gamma = q^{(\gamma, \alpha_i)} v_\gamma \qquad \gamma \in \Phi, \ i \in \{1,2\}$$

$$K_{\alpha_i} h_{\alpha_j} = h_{\alpha_j} \qquad i,j \in \{1,2\}$$

$$E_{\alpha_i} v_\gamma = \begin{cases} v_{\gamma+\alpha_i} & \text{if } \langle \gamma, \alpha_i^\vee \rangle = -1 \\ h_{\alpha_i} & \text{if } \gamma = -\alpha_i \\ 0 & \text{o/w} \end{cases}$$

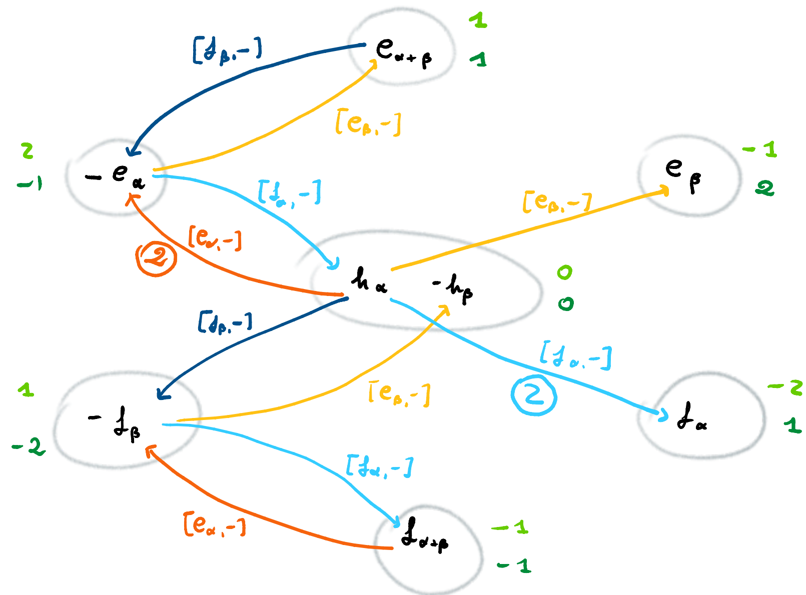
$$F_{\alpha_i} v_\gamma = \begin{cases} v_{\gamma-\alpha_i} & \text{if } \langle \gamma, \alpha_i^\vee \rangle = 1 \\ h_{\alpha_i} & \text{if } \gamma = \alpha_i \\ 0 & \text{o/w} \end{cases}$$

$$E_{\alpha_i} h_{\alpha_j} = \begin{cases} [2]\, v_{\alpha_i} = (q+q^{-1})\, v_{\alpha_i} & \text{if } i=j \\ v_{\alpha_i} & \text{if } i \neq j \end{cases}$$

$$F_{\alpha_i} h_{\alpha_j} = \begin{cases} (q+q^{-1})\, v_{-\alpha_i} & \text{if } i=j \\ v_{-\alpha_i} & \text{if } i \neq j \end{cases}$$

(Again one can check that this defines a U-module)

We can compare this to the classical adjoint representation of $\mathfrak{sl}_3$

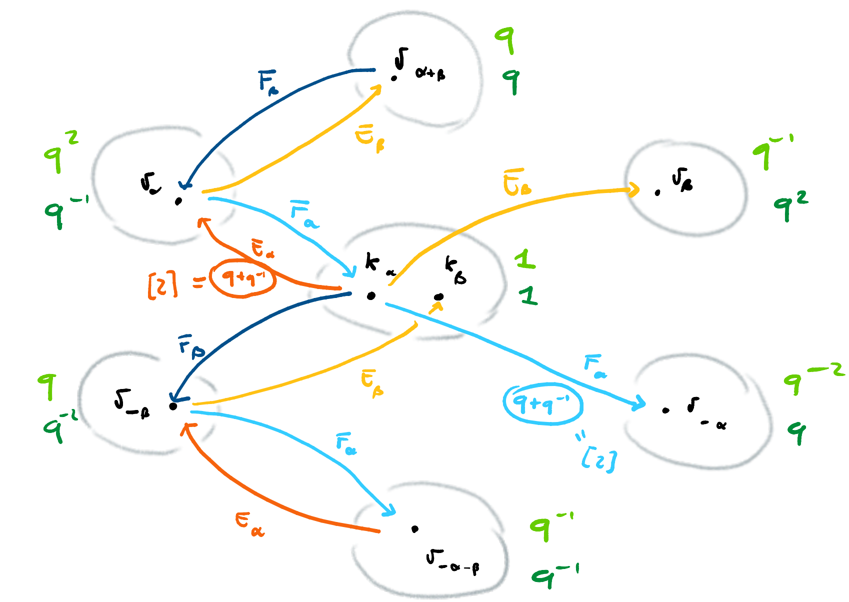


classical adjoint representation of type A_2

$= [h_\alpha, -]$

$= [h_\beta, -]$

(here $\alpha = \alpha_1$, $\beta = \alpha_2$)



quantum adjoint representation of type A_2

$= K_\alpha \cdot -$

$= K_\beta \cdot -$

($\alpha = \alpha_1$, $\beta = \alpha_2$)

Theorem If $u \in U$ annihilates all finite dimensional U-modules, then $u = 0$.

Pf (idea) Suppose $u \neq 0$:

i) choose bases $\{x_i\}$ and $\{y_j\}$ of U^+ and U^- (resp.) and write
$$u = \sum_{i,\mu,j} a_{j\mu i}\, y_j K_\mu x_i$$
$\Rightarrow$ we can find $\nu_0 \in \mathbb{Z}\Phi$ maximal s.t. $\exists\, j,\mu,i$ with $x_i \in U^+_{\nu_0}$ and $a_{j,\mu,i} \neq 0$

ii) Consider modules of the form
$$\tilde{L}(\lambda) \otimes {}^{\omega}\tilde{L}(\lambda') \qquad (\lambda, \lambda' \in \Lambda)$$
(where ${}^{\omega}(-)$ denotes the twist of a U-module via ω)

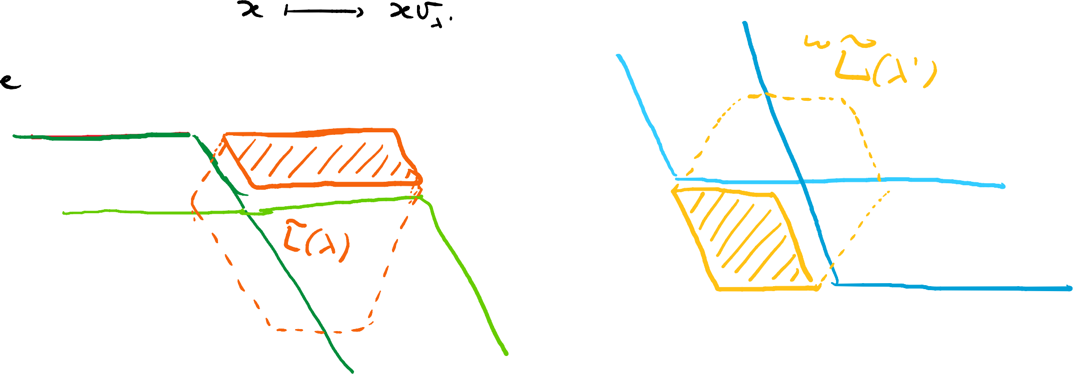
iii) Apply u to highest wt vectors $v_\lambda \otimes v'_{\lambda'}$ and project to the subspace $\tilde{L}(\lambda) \otimes ({}^{\omega}\tilde{L}(\lambda'))_{-\lambda'+\nu_0}$

Using the definition of Δ and the choice of ν_0 one can show, after some work, that this projection is:
$$\sum_{\substack{j,\mu,i \\ \deg(x_i)=\nu_0}} a_{j,\mu,i}\, q^{(\nu_0,\lambda)+(\mu,\lambda-\lambda'+\nu_0)-(\deg y_j,\, -\lambda'+\nu_0)}\, y_j v_\lambda \otimes x_i v'_{\lambda'} \qquad (*)$$
and by hypothesis this is 0 for all $\lambda, \lambda' \in \Lambda$

iv) We can find $N > 0$ s.t. $\nu_0 < \sum_{\alpha \in \Pi} N\alpha$ and $\nu'(j) < \sum_{\alpha\in\Pi} N\alpha$ for all j appearing in the sum $(*)$.

Let $\Lambda^+_N = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle > N \;\; \forall \alpha \in \Pi\}$.

v) Notice that, for $\lambda, \lambda' \in \Lambda^+_N$, the maps
$$U^-_{\nu'(j)} \to \tilde{L}(\lambda)_{\lambda-\nu'(j)}, \quad y \mapsto y v_\lambda \qquad \text{and} \qquad U^+_{\nu_0} \to {}^{\omega}\tilde{L}(\lambda')_{-\lambda'+\nu_0}, \quad x \mapsto x v'_{\lambda'}$$
are bijective



So the $y_j v_\lambda$ and the $x_i v'_{\lambda'}$ are linearly independent

vi) Hence the vanishing of $(*)$ implies

$$0 = \sum_{\mu \in \mathbb{Z}\Phi} a_{j,\mu,i} \, q^{(\nu_0,\lambda) + (\mu, \lambda - \lambda' + \nu_0) - (\nu'(j), -\lambda' + \nu_0)}$$

$$\Rightarrow \sum_{\mu \in \mathbb{Z}\Phi} a_{j\mu i} \, q^{(\mu, \nu_0 - \lambda')} \, q^{(\mu,\lambda)} = 0$$

for any i, j w/ $x_i \in U^+_{\nu_0}$

vii) To conclude we apply Artin's theorem on linear independence of characters:

<u>Lemma</u> S semi-group. $\chi_i : S \to k^\times$, $i \in I$ distinct characters (homomorphisms of semi-groups)
$\Rightarrow$ χ_i are linearly independent as functions $S \to k$

Here we take $S = \Lambda^+_N$, and the characters

$$\Lambda^+_N \longrightarrow k^\times \qquad \lambda \longmapsto q^{(\mu,\lambda)}$$

Then $a_{j,\mu i} \, q^{(\mu, \lambda' - \nu_0)} = 0 \Rightarrow a_{j,\mu,i} = 0$ which is a contradiction

$\square$

The characteristic 0 case

Theorem Suppose $char(k)=0$, $q \in k^\times$ not a root of unity

Then $\tilde{L}(\lambda) = L(\lambda)$, and the $\dim L(\lambda)_\mu$'s are given by Weyl's character formula

Pf (only in the case q is transcendental over $\mathbb{Q}$)

1) We deal first with the case $k = \mathbb{Q}(v)$

Consider $A = \mathbb{Q}[v, v^{-1}]$ and the A-submodules $L(\lambda)_A$ and $\tilde{L}(\lambda)_A$ (of $L(\lambda)$ and $\tilde{L}(\lambda)$ resp.) generated by all vectors of the form

$$F_{\alpha_1} F_{\alpha_2} \cdots F_{\alpha_k} v_\lambda \quad , \text{for } \alpha_i \in \Pi$$

(only finitely many of them are $\neq 0$)

and let $L(\lambda)_{A,\mu}$ (and $\tilde{L}(\lambda)_{A,\mu}$) be the submodules generated by all vectors as above with $\lambda - \alpha_1 - \cdots - \alpha_k = \mu$

These are free A-modules (they are f.g. and torsion free, and A is a PID), and we have

$$\mathrm{rk}_A \, L(\lambda)_{A,\mu} = \dim_{\mathbb{Q}(v)} L(\lambda)_\mu \quad \text{and}$$

$$\mathrm{rk}_A \, \tilde{L}(\lambda)_{A,\mu} = \dim_{\mathbb{Q}(v)} \tilde{L}(\lambda)_\mu$$

Now consider $\mathcal{A} = \mathbb{Q}[v, v^{-1}] \longrightarrow \mathbb{C}$

$$v \longmapsto 1$$

and denote $L(\lambda)_{\mathbb{C}} := L(\lambda)_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$

$$\tilde{L}(\lambda)_{\mathbb{C}} := \tilde{L}(\lambda)_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$$

Then $L(\lambda)_{\mathbb{C}}$ and $\tilde{L}(\lambda)_{\mathbb{C}}$ are f.d. modules for the complex Lie algebra $\mathfrak{g}$

Furthermore one can show:

1) The wt spaces are given by

$$(L(\lambda)_{\mathbb{C}})_{\mu} = L(\lambda)_{\mathcal{A},\mu} \otimes_{\mathcal{A}} \mathbb{C}$$

$$\Rightarrow \dim (L(\lambda)_{\mathbb{C}})_{\mu} = \mathrm{rk}_{\mathcal{A}} L(\lambda)_{\mathcal{A},\mu} = \dim_{\mathbb{Q}(v)} L(\lambda)_{\mu}$$

$$(\tilde{L}(\lambda)_{\mathbb{C}})_{\mu} = \tilde{L}(\lambda)_{\mathcal{A},\mu} \otimes_{\mathcal{A}} \mathbb{C} \quad \Rightarrow \quad \cdots$$

2) $L(\lambda)_{\mathbb{C}}$ and $\tilde{L}(\lambda)_{\mathbb{C}}$ are simple

Hence we get the claim from the classical theory.

2) by transcendence of q we get

$$\mathbb{Q}(v) \underset{v \mapsto q}{\hookrightarrow} k$$

We have isomorphisms

$$M(\lambda)_k \xrightarrow{\sim} M(\lambda)_{\mathbb{Q}(v)} \underset{\mathbb{Q}(v)}{\otimes} k$$

$$\widetilde{L}(\lambda)_k \xrightarrow{\sim} \widetilde{L}(\lambda)_{\mathbb{Q}(v)} \underset{\mathbb{Q}(v)}{\otimes} k$$

$$L(\lambda)_k \xrightarrow{\sim} L(\lambda)_{\mathbb{Q}(v)} \underset{\mathbb{Q}(v)}{\otimes} k$$

So the claim follows from case (1) $\square$

Let us state two other results from classical theory that apply also in this setting:

<u>Proposition</u> Let $\lambda \in \Lambda^+$, then $L(\lambda)^* \cong L(-w_0\lambda)$

<u>Theorem</u> Suppose $\mathrm{char}(k) = 0$, then any finite dimensional U-module is semisimple.

pf

It is enough to show that any short exact sequence of U-modules

$$0 \longrightarrow L(\lambda) \xrightarrow{\iota} M \xrightarrow{\pi} L(\mu) \longrightarrow 0$$

splits

Case 1 $\mu \not< \lambda$

Pick $\bar{v} \in L(\mu)_\mu \setminus \{0\}$, $\exists\, v \in M_\mu$ s.t. $\pi(v) = \bar{v}$

We have $E_\alpha v = 0 \quad \forall \alpha \in \Pi$, o/w $\mu + \alpha$ would be a wt of M and hence of $L(\lambda)$ (b/c it cannot be a wt of $L(\mu)$) but this contradicts the assumption

So Uv is a f.d. quotient of $M(\mu)$, hence of $\tilde{L}(\mu)$ (by Lemma (*), b) which is $\cong L(\mu)$ by the theorem

So $Uv \cong L(\mu)$ and $\ker \pi \cap Uv = \{0\}$

$\Rightarrow \ker \pi \oplus Uv = M$

Case 2 $\mu < \lambda$

Dualizing the sequence we get

$$0 \longrightarrow L(\mu)^* \longrightarrow M^* \longrightarrow L(\lambda)^* \longrightarrow 0$$

which by the proposition becomes

$$0 \longrightarrow L(-w_0\mu) \longrightarrow M^* \longrightarrow L(-w_0\lambda) \longrightarrow 0$$

Now $\mu < \lambda \Rightarrow -w_0\mu < -w_0\lambda \Rightarrow -w_0\lambda \not< -w_0\mu$

So we are in case 1 and $M^* \cong L(\mu)^* \oplus L(\lambda)^*$

$\Rightarrow M \cong (M^*)^* = L(\mu) \oplus L(\lambda)$ $\square$