MATH 353, HW1, DUE FEB 7

There are 2 problems worth 12 points total. Your score for this homework is the minimum of the sum of the points you've got and 10. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. Each problem is in its own section, which also features some discussion. You are responsible for establishing claims in the "Problem" sections (for this problem set, these are Sections 1.2 and 2.1). Note that Section 2.2 is not for credit.

As we haven't covered much theory yet, the purpose of this problem set is to study representations using elementary (e.g., Linear Algebra) means – and also to gain some intuition into what the representations are. The problem set discusses arbitrary representations of cyclic groups as well as irreducible representations of the "binary dihedral group" (that will be introduced in the corresponding section).

1. Representations of cyclic groups

Let $m > 1$ be an integer, and consider the cyclic group $G = \mathbb{Z}/m\mathbb{Z}$. The purpose of this problem is to describe the finite dimensional representations of G over an algebraically closed field $\mathbb{F}$. For this we need to recall the Jordan normal form (JNF) theorem from Linear Algebra.

1.1. **JNF Theorem.** We start by recalling the notions of Jordan blocks and Jordan matrices. Let n be a positive integer and $\lambda \in \mathbb{F}$. Let I_n denote the identity matrix of size n and J_n denote the matrix with 1's right above the main diagonal and zeroes everywhere else, e.g., $J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. A *Jordan block* $J_n(\lambda)$, by definition, is $\lambda I_n + J_n$. For example, $J_3(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. A *Jordan matrix* is a block-diagonal matrix

$$\mathrm{diag}(J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k))$$

for some positive integers $n_1, \ldots, n_k$ and some elements $\lambda_1, \ldots, \lambda_k$. For example, the following is an example of a Jordan matrix:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \lambda, \mu \in \mathbb{F}.$$

This class of matrices is important thanks to the following theorem.

Theorem 1.1. *For every linear operator on a finite dimensional vector space over $\mathbb{F}$ there is a basis in which it is presented by a Jordan matrix. Moreover, this matrix is uniquely determined up to permuting the blocks.*

And now we proceed to understanding the representations of G using this theorem.

1.2. **Problem, 4pts.** a, 1pt) Let V be a finite dimensional vector space over $\mathbb{F}$. Show that there is a bijection between

- Representations of G in V (up to isomorphism of representations),
- and conjugacy classes linear operators on V of *order dividing* m (meaning operators A with $A^m = \mathrm{Id}_V$), where, recall $G = \mathbb{Z}/m\mathbb{Z}$.

that depends on the choice of a generator in G.

b, 2pt) Suppose $\mathbb{F}$ is of characteristic 0 (e.g. $\mathbb{C}$). Let V be a finite dimensional representation of G. Show that V is isomorphic to the direct sum of 1-dimensional representations, each of the following form: if we view G as the group of mth roots of 1, then the one-dimensional representations are given by $g \mapsto g^k$ for some $k = 0, \ldots, m-1$. *Hint: you need to use the JNF theorem and the binomial formula.*

c, 1pt) Give a counterexample to the direct analog of the claim in b) when $\mathbb{F}$ has positive characteristic.

2. Representations of binary dihedral groups

Let n be a positive integer. Let G be the following subset of $\mathrm{GL}_2(\mathbb{C})$:

$$G = \{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} | \epsilon^{2n} = 1\}.$$

It is easy to see that G is a subgroup of order $4n$. This group is known as the *binary dihedral group.* The goal of this problem is to classify its finite dimensional irreducible representations over $\mathbb{C}$ (the same classification will work over any algebraically closed characteristic 0 field).

To start with this, consider the following two elements of G:

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, t = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$$

for a fixed primitive root of unity ϵ of order $2n$.

Also note that the subgroup of G consisting of diagonal matrices, to be denoted by H, is a cyclic group of order $2n$. Essentially, our main tool to classify the irreducible representations of G is to study their restrictions to H (whose representations we know thanks to the previous problem) – compare to Remark in Section 2.4 of Lecture 2.

2.1. **Problem, 8pts total.** a, 1pt) Show that every element of G is uniquely written as either t^m or st^m with $m = 0, 1, \ldots, 2n-1$ with multiplication recovered from the relations $e = t^{2n}, s^2 = t^n, sts^{-1} = t^{-1}$.

b,1pt) Deduce that there is a bijection between

- Representations of G in a finite dimensional vector space V (up to isomorphism),
- and pairs of operators $S, T \in \mathrm{GL}(V)$ satisfying $S^2 = T^n, T^{2n} = \mathrm{Id}, STS^{-1} = T^{-1}$ (up to simultaneous conjugation).

c, 1pt) Show that the assignments sending t to $z := \pm 1$ and s to $\pm\sqrt{z^n}$ define four pairwise non-isomorphic 1-dimensional representations of G. Furthermore, show that every 1-dimensional representation is isomorphic to one of these.

d, 1pt) Let V be a finite dimensional representation of G and v be an eigenvector for t with eigenvalue λ. Check that sv is also an eigenvector for t, now with eigenvalue λ^{-1}.

e, 1pt) In the notation of the previous part, show that $\mathrm{Span}_{\mathbb{C}}(v, sv)$ is a subrepresentation in the representation V of G. Deduce that every finite dimensional irreducible representation of G has dimension at most 2.

f, 1pt) Prove that if V is a 2-dimensional irreducible representation of G, then λ from d) is not equal to ± 1.

g, 1pt) Prove that V is uniquely recovered from $\lambda (\neq \pm 1)$ up to an isomorphism, and, moreover, the irreducible representations corresponding to λ and λ^{-1} are isomorphic. *Hint: try to recover the matrices corresponding to s, t in a suitable basis.*

h, 1pt) Conclude that G has exactly 4 one-dimensional (irreducible) representations and $n-1$ two-dimensional irreducible representations.

2.2. **Not for credit.** This part examines a special case of the so called McKay correspondence. One point is: we'll do something weird to get something nice (as often happens in Mathematics). Another point is to practice tensor products of group representations. We assume the Maschke theorem: representations of finite groups over $\mathbb{C}$ are completely reducible (equivalently, are direct sums of irreducibles).

Note that the inclusion $G \subset \mathrm{GL}_2(\mathbb{C})$ gives rise to the distinguished 2-dimensional representation of G (by restriction) to be denoted simply by $\mathbb{C}^2$. Let $V_0, \ldots, V_r$ be the irreducible representations of G with V_0 being the trivial representation (so that $r = n+2$). We construct an un-oriented graph with vertices $0, \ldots, r$, where i and j are connected by edges whose number is the multiplicity of V_i in $\mathbb{C}^2 \otimes V_j$.

a) Show that this is well-defined: the multiplicity of V_i in $\mathbb{C}^2 \otimes V_j$ is the same as the multiplicity of V_j in $\mathbb{C}^2 \otimes V_i$. *Hint: observe that $\mathbb{C}^2 \cong (\mathbb{C}^2)^*$ as representations of G and use the "tensor-Hom adjunction" together with the Schur Lemma.*

b) Draw the resulting graph. *Hint: restrict to H.*

c) Google "Dynkin diagrams". What you get in b) is the "Dynkin diagram of affine type D_n" a.k.a. the "extended Dynkin diagram of type D_n". If one removes the node 0, one gets the "Dynkin diagram of type D_n". And if you were to do this construction with the group H (very easy), you would get the "Dynkin diagram of affine type A_{2n}". We'll see the subgroup of $\mathrm{SL}_2(\mathbb{C})$ serving "E_6" later.