MATH 353, HW2, DUE FEB 21

There are 3 problems worth 13 points total. Your score for this homework is the minimum of the sum of the points you've got and 10. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. Each problem is in its own section, which also features some discussion. You are responsible for establishing claims in Section 1, 2.2 and 3. Note that Section 1.1 is not for credit.

This problem set is on the topic "Irreducible and completely reducible representations", from Lecture 5 to the first section of Lecture 8.

1. Irreducible representations inside polynomials, 4pts

Let $\mathbb{F}$ be a characteristic 0 field. Suppose that G is a finite group and V is its *faithful* finite dimensional representation (meaning that the corresponding homomorphism $G \to \mathrm{GL}(V)$ is injective). Let $\mathbb{F}[V]$ denote the algebra of polynomials on V. The goal of this problem is to show that

(*) every irreducible representation of G occurs as a direct summand in $\mathbb{F}[V]$.

a, 1pt) Prove that there is $v \in V$ with trivial stabilizer in G.

b, 1pt) Consider the map $\mathbb{F}[V] \to \mathrm{Fun}(Gv, \mathbb{F})$ given by restricting a polynomial function to Gv. Show that it is surjective.

c, 2pts) Prove (*) above.

1.1. **Not for credit.**

Here is an application of (*). Suppose that we have a family $\mathcal{F}$ of representations of G containing the trivial one and closed under the operations of taking direct sums and summands, tensor products, and duals (but duals can be removed). Show that there is a unique normal subgroup $H_{\mathcal{F}}$ (depending on $\mathcal{F}$) such that a representation is in $\mathcal{F}$ if and only if $H_{\mathcal{F}}$ acts trivially (by the identity endomorphism) on that representation.

2. Going from real to complex

The goal of this problem is to understand the behavior of the *complexifications* of finite dimensional irreducible representations over $\mathbb{R}$ by "changing the base to $\mathbb{C}$".

2.1. **Background.**

You do not need to prove the claims in this section.

Let V be a vector space over $\mathbb{R}$. To V we can assign a vector space $V_{\mathbb{C}}$ over $\mathbb{C}$ via $V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$. By what was explained in Lecture 4, this is an $\mathbb{R}$-vector space. Then $V_{\mathbb{C}}$ has the unique vector space structure over $\mathbb{C}$ such that $a(z \otimes v) = (az) \otimes v$ for all $v \in V, a, z \in \mathbb{C}$. It is easy to see that, if A is an $\mathbb{R}$-algebra, then $A_{\mathbb{C}}$ has a natural structure of a $\mathbb{C}$-algebra (somewhat informally, it has the same basis as A with the same mutliplication – but now is a space over $\mathbb{C}$). Similarly, if V is an A-module, then $V_{\mathbb{C}}$ is naturally an $A_{\mathbb{C}}$-module.

Below you can also use a fact mentioned in the lectures: that the only finite dimensional algebras over $\mathbb{R}$ which are skew-fields are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, of dimensions $1, 2, 4$, respectively.

2.2. **Problem itself, 6pts.** a, 2pts) Let A be an associative algebra over $\mathbb{R}$, and let U, V be A-modules. Construct an isomorphism of vector space over $\mathbb{C}$:

$$\mathbb{C} \otimes_{\mathbb{R}} \operatorname{Hom}_A(U, V) \xrightarrow{\sim} \operatorname{Hom}_{A_{\mathbb{C}}}(U_{\mathbb{C}}, V_{\mathbb{C}}).$$

Moreover, for $U = V$, prove that you get an algebra isomorphism.

b, 2pts) Now suppose V is a finite dimensional irreducible A-module. Prove that we have exactly one of the following options hold:

($\mathbb{R}$) $\operatorname{End}_A(V) \cong \mathbb{R}$, in which case $V_{\mathbb{C}}$ is irreducible over $A_{\mathbb{C}}$.

($\mathbb{C}$) $\operatorname{End}_A(V) \cong \mathbb{C}$, in which case $V_{\mathbb{C}}$ splits as the direct sum of two non-isomorphic irreducible $A_{\mathbb{C}}$-modules.

($\mathbb{H}$) $\operatorname{End}_A(V) \cong \mathbb{H}$, in which case $V_{\mathbb{C}}$ splits as the direct sum of two isomorphic irreducible $A_{\mathbb{C}}$-modules.

c, 1pt) Give an example of a 2-dimensional irreducible representation of $\mathbb{Z}/4\mathbb{Z}$ over $\mathbb{R}$ satisfying ($\mathbb{C}$). *Hint: this example should have something to do with the complex numbers.*

d, 1pt) Give an example of a 4-dimensional irreducible representation of the binary dihedral group with 8 elements (Problem 2 in HW1) over $\mathbb{R}$ satisfying ($\mathbb{H}$). *Hint: and now with the quaternions.*

3. Irreducible representations of p-groups in characteristic p, 3pts

Let $\mathbb{F}$ be an algebraically closed field of characteristic p and G be a finite p-group (i.e., a group whose order is p^k for some k). Show that the only finite dimensional irreducible representation of G over $\mathbb{F}$ is the trivial one. *Hint: use that G has nontrivial center, and use some results from Lecture 8 – with some other arguments – to show that all elements in the center act by the identity endomorphism on every irreducible representation.*