MATH 353, HW3, DUE MAR 30

There are 3 problems worth 12 points total. Your score for this homework is the minimum of the sum of the points you've got and 10. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. Each problem is in its own section, which also features some discussion. You are responsible for Sections 1,2 and 3.2.

This problem set is on the topic "Characters", from the 2nd section of Lecture 8 to the first two sections of Lecture 11.

1. $\mathrm{Fun}(G,\mathbb{C})$ as a representation of $G\times G$, 3pts

The group $G\times G$ acts on G by $(g_1,g_2).g := g_1 g g_2^{-1}$. Hence $V := \mathrm{Fun}(G,\mathbb{C})$ is a representation of $G\times G$. The goal of this problem is to compute the character of this representation and decompose it into the direct sum of irreducible representations.

1, 1pt) Prove that $\chi_V(g_1,g_2)=0$ if $g_1,g_2\in G$ are not conjugate, and $\chi_V(g_1,g_2)$ equals to the order of the centralizer $Z_G(g_1)$ if g_1,g_2 are conjugate.

2, 2pts) Let $U_1,\ldots,U_k$ be all (pairwise non-isomorphic) irreducible representations of G. Prove the following isomorphisms of representations of $G\times G$:

$$V\cong\bigoplus_{i=1}^k U_i\otimes U_i^*,$$

where the structure of a representation of $G\times G$ on $U_i\otimes U_i^*$ is as in Sec. 2 of Lecture 11.

2. Irreducible representations of A_4, 3pts

This problem classifies the irreducible representations of the alternating group A_4 (the even permutations in S_4).

1, 2pts) Use characters to show that the restriction of the representation $\mathbb{C}^4_0$ of S_4 to A_4 is irreducible, while the restriction of V_2 splits into the sum of two pairwise non-isomorphic one-dimensional non-trivial representations.

2, 1pt) Classify the irreducible representations of A_4, the conjugacy classes of A_4, and write down the character table.

3. Irreducible representations of the binary tetrahedral group

3.1. **Setting.** Consider the unit quaternion group $H:=\{\pm1,\pm i,\pm j,\pm k\}$ (with product as in $\mathbb{H}$), isomorphic to the binary dihedral group with 8 elements. The group $K:=\mathbb{Z}/3\mathbb{Z}$ acts on H by automorphisms, a generator σ of K sends $\pm i$ to $\pm j$, $\pm j$ to $\pm k$ and $\pm k$ to $\pm i$ (preserving the sign) and fixes 1 (automatic) and -1. The *binary tetrahedral group* can be realized as the semi-direct product $K\ltimes H$ (H is normal). Note that $G/\{\pm1\}$ is identified with A_4 (and under this identification, $H/\{\pm1\}$ is identified with the Klein four-group).

3.2. **Problem, 6pts.** Our goal is to classify the irreducible representations of G, compute their characters, and decompose tensor products with a certain 2-dimensional representation.

1, 2pts) Prove that there are 7 irreducible representations of G total. In more detail, prove that four of them are pulled back from the irreducible representations of A_4 (under the epimorphism explained in the previous section), three of dimension 1 and one of dimension 3. The other three are all 2-dimensional, where the element $-1 \in H \subset G$ acts by the scalar -1. *Hint: prove that the element $-1 \in G$ must act on a 1-dimensional representation of G by* 1.

2, 1pt) Describe the conjugacy classes in G (explain which elements of G viewed as the semidirect product lie in the each of conjugacy classes).

3, 2pts) This is harder... Write down the character table. You are allowed to use the following fact: for one of the 2-dimensional irreducible representations, the image of the corresponding homomorphism $G \to \mathrm{GL}_2(\mathbb{C})$ lies in $\mathrm{SL}_2(\mathbb{C})$. We will denote this representation by $\mathbb{C}^2$.

4, 1pt) For each of the seven irreducible representations of G, decompose its tensor product with the representation $\mathbb{C}^2$ from 3) into the direct sum of irreducibles.

3.3. **Extra credit.** 5) Record this as a graph as in the extra credit problem in homework 1. Google Dynkin diagrams and recognize your answer as the Dynkin diagram of affine type E_6 (when you remove the node corresponding to the trivial representation you get the Dynkin diagram of type E_6).

6) Let V be a finite dimensional complex vector space. By a *complex reflection* in $\mathrm{GL}(V)$ we mean a finite order element s such that $\mathrm{rk}(s - \mathrm{id}_V) = 1$. Let U be one of the two 2-dimensional irreducible representations different from $\mathbb{C}^2$. Show that the image of G in $\mathrm{GL}(U)$ is generated by complex reflections (in a future bonus lecture on why we consider groups that appear in our examples we will explain why such subgroups in the general linear group are important).