MATH 353, HW4, DUE APR 13

There are 3 problems worth 12 points total. Your score for this homework is the minimum of the sum of the points you've got and 10. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. Each problem is in its own section, which also features some discussion. The problems themselves are in Sections 1, 2.2, 3.1.

This problem set is on the topics of induction and representations of symmetric groups, from Lecture 14 to Lecture 18.

1. Transitivity of induction via characters, 2pts

Suppose that $\mathbb{F}$ is a characteristic 0 field. Let $K \subset H \subset G$ be finite groups, and let W be a representation of K. Use the Frobenius character formula for induced representations to deduce that the characters of $\operatorname{Ind}_K^G W$ and $\operatorname{Ind}_H^G \operatorname{Ind}_K^H W$ coincide. *Note that these representations are naturally isomorphic for any $\mathbb{F}$ but your proof has to use the Frobenius formula from Lecture 15.*

2. From representations of S_n to representations of A_n

The goal of this problem is to deduce the classification of the irreducible representations of A_n from those of S_n. Here $\mathbb{F}$ is an algebraically closed field of characteristic 0.

2.1. **Twists of representations.** For this we need a general construction. Let $H \subset G$ be finite groups, and suppose H is normal. Let U be a representation of H and let $\rho : H \to \mathrm{GL}(U)$ be the corresponding homomorphism. For $g \in G$, define a new representation U^g of H, by definition, it corresponds to the homomorphism $h \mapsto \rho(ghg^{-1})$. One can show (you are encouraged to show this, but this is not for credit) that, up to an isomorphism of representations of H, the representation U^g only depends on the coset gH.

2.2. **Problem, 6pts.** Set $G = S_n, H = A_n$. Prove the following claims (*hint: some of them require Frobenius reciprocity*).

a, 1pt) For a finite dimensional representation U of H we have an isomorphism of representations of H:

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G U \cong U \oplus U^g,$$

for any odd permutation $g \in G$.

b, 1pt) For a finite dimensional representation V of G we have an isomorphism of representations of G

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G V \cong V \oplus \operatorname{sgn} \otimes V.$$

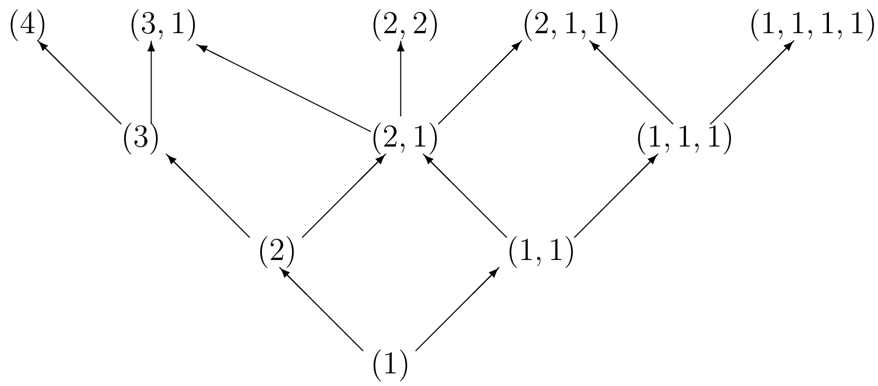
c, 2pts) Let λ be a partition of n and V_λ denote the corresponding irreducible representation of G. Prove that $\operatorname{Res}_H^G V_\lambda$ is irreducible if $\lambda \neq \lambda^t$ and splits into the direct sum of two pairwise non-isomorphic irreducibles else.

d, 1pt) Let U_λ denote $\operatorname{Res}^G_H V_\lambda$ if $\lambda \neq \lambda^t$. For $\lambda = \lambda^t$, let U_λ^+, U_λ^- denote the irreducible summands of $\operatorname{Res}^G_H V_\lambda$. Show that for $\lambda \neq \mu$, the representation $U_\lambda^\pm$ is not isomorphic to either $U_\mu^\pm$ or U_μ, while U_λ is isomorphic to U_μ if and only if $\lambda = \mu^t$.

e, 1pt) Prove that every irreducible representation of H is isomorphic to U_λ or $U_\lambda^\pm$. *This finishes the classification of irreducible representations of H.*

3. Restriction to S_{n-m}

The goals of this problem is to study the restrictions of the irreducible representations V_λ of S_n first to S_{n-1}, then to S_{n-m} for arbitrary, and finally giving a combinatorial labelling set for a basis in V_λ. For this we need the notion of the *Young graph.* The vertices in this oriented graph are Young diagrams, with one edge from μ to λ if μ is obtained from λ by removing a single box. On the level of partitions, this means that there is an index i such that $\mu_j = \lambda_j$ for $j \neq i$, while $\lambda_i > \lambda_{i+1}$ and $\mu_i = \lambda_i - 1$. Here is a fragment of the Young graph (where we use partitions instead of Young diagrams)



3.1. **Problem, 4pts.** a, 2pts) Use the Frobenius formula for the character of V_λ (from Lecture 17) to show that $\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda$ decomposes into the direct sum of irreducible representations of S_{n-1} as follows: it is the direct sum of V_μ, where a diagram μ with $n-1$ boxes is obtained from λ by removing a box, each occurring with multiplicity 1.

b, 1pt) Let λ be a partition of n. Show that $\operatorname{Res}^{S_n}_{S_{n-m}} V_\lambda$ decomposes into the direct sum of irreducible representations of S_{n-m} indexed by the length m paths in the Young graph that end at λ, and that the irreducible corresponding to any given path is labelled by its starting vertex.

c, 1pt) Establish a basis in V_λ (defined up to rescaling each of the individual basis vectors) indexed by paths from (1) to λ.

3.2. **Discussion.** A path from (1) to λ can be alternatively presented as a *standard Young tableaux* on λ. By definition, it is a filling of the Young diagram λ with numbers from 1 to n (where n is the number of boxes in λ) such that the numbers increase left to right and bottom to top. To produce such a bijection is a not for credit exercise.