MATH 353, HW5, DUE APR 27

There are 3 problems worth 12 points total. Your score for this homework is the minimum of the sum of the points you've got and 10. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. Each problem is in its own section, which also features some discussion. The problems themselves are in Sections 1.1, 2.1, and 3.1.

This problem set is on the structure and representations of semisimple associative algebras, Lectures 19-23.

1. Endomorphisms

Let A be an associative algebra over a field $\mathbb{F}$, V be its finite dimensional completely reducible representation. From Lecture 21 we know that there is a natural isomorphism

$$\bigoplus_{i=1}^{k} U_i \otimes_{S_i} M_i \xrightarrow{\sim} V,$$

where $U_1, \ldots, U_k$ are pairwise non-isomorphic irreducible A-modules, $S_i := \operatorname{End}_A(U)^{opp}$, and $M_i := \operatorname{Hom}_A(U_i, V)$.

1.1. **Problem, 4pts.** a, 2pts) Establish a natural isomorphism

$$\bigoplus_{i=1}^{k} \operatorname{End}_{S_i}(M_i) \xrightarrow{\sim} \operatorname{End}_A(V)$$

Hint: let S be a skew-field, U a right S-module, M a left S-module and $\varphi \in \operatorname{End}_S(M)$. Show that there is a unique linear operator on the $\mathbb{F}$-vector space $U \otimes_S M$ that sends $u \otimes m$ to $u \otimes \varphi(m)$ for all $u \in U, m \in M$.

b, 2pts) Set $B := \operatorname{End}_A(V)$. Show that the image of A in $\operatorname{End}_{\mathbb{F}}(V)$ is $\operatorname{End}_B(V)$.

1.2. **Discussion.** Let us explain a moral application of a). Suppose that G is a (finite, for simplicity) group and V be its (finite dimensional, for simplicity) representation over $\mathbb{C}$ equipped with an invariant Hermitian scalar product. Let $H : V \to V$ be a Hermitian operator and a G-equivariant map (i.e., $H \in \operatorname{End}_G(V)$). Suppose we want to diagonalize H. It turns out that this reduces to diagonalizing the images of H in $\operatorname{End}_{\mathbb{C}}(M_i)$ (these spaces turn out to have Hermitian scalar products induced from V).

This is a baby example. A more interesting situation, where one proceeds along the same lines is when V is an infinite dimensional Hilbert space, G is a Lie group acting on V by unitary operators (with some continuity conditions). In Quantum Mechanics, a Hamiltonian is a Hermitian operator on V and to solve the corresponding system is to diagonalize H. In many situations, the multiplicity spaces are finite dimensional (and, if we are especially lucky, they are 1-dimensional). Then one can reduce the diagonalization problem to the finite dimensional case.

2. Centralizer

Here we assume that the base field $\mathbb{F}$ is algebraically closed. Let B be a semisimple finite dimensional $\mathbb{F}$-algebra, and A be some associative algebra equipped with a homomorphism $\varphi : A \to B$. Define the *centralizer algebra* $Z_A(B)$ consisting of all $b \in B$ such that $b\varphi(a) = \varphi(a)b$ for all $a \in A$ (it is a subalgebra in B, but you don't need to check that).

2.1. **Problem, 3pts.** a, 2pts) Suppose that A is also finite dimensional and semisimple. Show that the following two conditions are equivalent:

- $Z_A(B)$ is commutative.
- For every irreducible B-module V and every irreducible A-module U, the multiplicity of U in V (viewed as an A-module) is equal to 0 or 1.

b, 1pt) Suppose that $\mathbb{F}$ is algebraically closed and of characteristic 0. Let $B = \mathbb{F}S_n$ and A be its subalgebra $\mathbb{F}S_{n-1}$ (and the homomorphism φ is the inclusion). Show that $Z_A(B)$ is commutative.

2.2. **Comment.** It turns out that one can prove b) directly. More precisely, $Z_A(B)$ is generated by the center of $\mathbb{F}S_n$ (or center of $\mathbb{F}S_{n-1}$) and the n-th *Jucys-Murphy* element $J_n := \sum_{i=1}^{n-1}(i,n)$. This observation is a starting point for the Okounkov-Vershik approach (discovered around 1996) to representations of symmetric groups.

3. Characters and irreducibles in positive characteristic

For this problem, the base field $\mathbb{F}$ is algebraically closed. We know that if $\operatorname{char}\mathbb{F} = 0$, then the number of irreducible $\mathbb{F}G$-modules equals to the number of conjugacy classes in G. It turns out that this can be generalized to the case when $\mathbb{F}$ has characteristic p: the number of irreducible $\mathbb{F}G$-modules is equal to the number of conjgacy classes in G whose elements have order coprime to p. The goal of this problem is to prove the upper bound on the number of irreducibles.

Let A be an associative algebra over an algebraically closed field $\mathbb{F}$. Let V be its finite dimensional module. Define the *character* $\chi_V : A \to \mathbb{F}$ by $\chi_V(a) = \operatorname{tr}(a_V)$.

3.1. **Problem, 5pts.** a, 2pts) Prove that the characters of different finite dimensional irreducible representations of A are linearly independent (as elements of $\operatorname{Hom}_{\mathbb{F}}(A, \mathbb{F})$). *Hint: reduce to the case of semisimple algebras.*

b, 1pt) From now, let G be a finite group, and assume that $\operatorname{char}\mathbb{F} = p > 0$. Let $g \in G$. Show that there are unique elements $g_s, g_u \in G$ such that

- g_s has order coprime to p,
- the order of g_u is a power of p,
- and $g = g_s g_u = g_u g_s$.

c, 1pt) Show that g and g_s have the same eigenvalues (with multiplicities) in any finite dimensional $\mathbb{F}G$-module V. Conclude that $\chi_V(g) = \chi_V(g_s)$.

d, 1pt) Prove that the number of irreducible $\mathbb{F}G$-modules does not exceed the number of conjgugacy classes of elements whose order is coprime to p.