IMPORTANT INFORMATION ON MATH 353

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Contents

1. What this class is about

Representation theory studies *representations* of groups, associative algebras (=associative rings with a compatible vector space structure over a given field), etc., i.e. homomorphisms from a given group/algebra to the *general linear group/algebra*. For groups, this will be the group of all invertible square matrices, or, more conceptually and more generally, the group of all invertible linear operators on a vector space. For associative algebras, we consider the algebra of all matrices (or all linear operators on a vector space).

This is an introductory course. We focus on representations with complex coefficients (or with coefficients in more general fields of characteristic 0, such as the reals) and look at representations of finite groups or of finite dimensional associative algebras (mostly assumed to be *semi-simple*). The cases of groups and of associative algebras will constitute two, larger and smaller, respectively, parts of the course.

The italized terms, such as in the previous paragraph, will not be explained in this document, but will be explained in the course.

It is worth noting that there is a lot of material that can be covered in such a course and some choices need to be made. Some of the material not covered in the main part of the course will be presented as "bonuses". Here are examples of things that are **not** going to appear (at least, in a systematic way):

- Infinite groups, e.g., Lie groups and related objects such as Lie algebras. My semi-educated guess (of a faculty working in the area and the Director of Graduate

Studies) is that there's going to be a 600 level graduate course on this topic in the next academic year having this course as well as MATH 526, Intro to differentiable manifolds, as prerequisites.
- Connections to Geometry. These connections are numerous (there's even the field of Geometric Representation theory which is my primary area of expertise) but are mostly beyond the scope of this course. We may touch upon things like reflection groups and symmetries of regular polyhedra.
- Connections to Physics, that are again numerous. Perhaps, most notably, so called unitary representations of groups play the role of symmetries in Quantum Mechanics. But this is again beyond the scope of this course.
- Modular Representation theory, i.e., the study of representations with coefficients in positive characteristic fields. This has been a very hot subject in the last decade but is, again, beyond the scope of this course.
- Et cetera...

The next two sections describe what I plan to cover.

2. Representations of finite groups

We will be interested in the following four, closely related questions:

- Given a finite group G, completely describe its finite dimensional representations,
- and learn how to compute their numerical invariants such as dimensions, or, even stronger, "characters".
- Applications of representations and characters to the structure theory of finite groups,
- and their connections to other parts of Mathematics.

Here is a preliminary plan for this part.

2.1. **Introduction.** I elaborate on what these bullets include.

2.2. **Basics.**

(1) Main definitions (of representations, homomorphisms, subrepresentations, etc.) and some examples.
(2) Representations of associative algebras. The group algebra.
(3) Tensor products of vector spaces and of group representations. Duals.

2.3. **Irreducible and completely reducible representations.** The irreducible representations are basic building blocks for more general representations. Completely reducible ones are those obtained from the irreducibles in the simplest possible way (for example, an irreducible representation itself is completely reducible, however paradoxical this may sound). In this part we study these classes of representations:

(1) Irreducible representations. The Schur lemma. A likely digression into skew-fields, in general, and the quaternions, in particular.
(2) Completely reducible representations and their basic properties and structure.
(3) Maschke's theorem: a finite dimensional representation of a finite group over the complex numbers is completely reducible.

2.4. **Characters.** Characters are functions on the group that, in a way, completely incorporate all numerical information about representations.

(1) The definition and basic properties of characters, examples.
(2) Orthogonality of characters and applications to the study of irreducible representations.

2.5. **Applications to the structure of finite groups.** Here we use characters to prove the Burnside $p^a q^b$ theorem: that a group of this order cannot be simple, i.e., must have a nontrivial normal subgroups. This is the first (but definitely not the last) example of how the representation theory applies to the structure theory of finite groups. Along the way, we will establish the Frobenius divisibility theorem, whose proof uses similar techniques. And we digress into algebraic integers.

2.6. **Induced representations.** The induction (from a subgroup to an ambient group) is a primary way of constructing representations. We will define the induction, relate it to the restriction of representations to a subgroup (the Frobenius reciprocity), and get a formula for the character of an induced representation, also due to Frobenius.

2.7. **The representation theory of symmetric groups.** The richest and most interesting part of the theory deals with understanding the irreducible representations of some concrete (families of) groups. The symmetric group S_n is interesting for many reasons (to be explained). In this course we classify the irreducible representations of symmetric groups and explain how to compute their characters via the Frobenius character formula. These results establish a connection between the representation theory of symmetric groups and classical objects of Algebraic Combinatorics such as partitions, Young tableaux, and symmetric polynomials. Energy permitting we will also discuss connections to Probability – as a bonus.

3. Representations of finite dimensional semisimple associative algebras

Here is a brief outline – more details will be posted later.

In this section we will be concerned with the representation theory of finite dimensional associative algebras. We will examine how the irreducible representations look like. We will introduce the class of *semisimple* algebras – those whose representations are completely reducible – and relate them to *simple* algebras – those without nontrivial two-sided ideals. We will then study the structure of finite dimensional simple algebras.

Time permitting we will also study the representations of algebras that are not semisimple, perhaps, concentrating on the path algebras of quivers.

4. References

Our main reference is "Introduction to Representation theory" by Etingof et al (Etingof was and is a Professor at MIT and his coauthors were high school students – or fresh out of high school – when the book was written). It is available from Etingof's MIT webpage. This an excellent book, clearly written, with many nice exercises. We are not going to cover all of it, but students are encouraged to read the remaining parts.

I expect to use two secondary references. The first is "A course in Algebra" by Vinberg – this is a textbook I used to learn Algebra when I was in college. I expect to follow some parts of that book here and there. The second book, "Young tableaux. With applications to representation theory and geometry" by Fulton is likely to be used when we talk about the representation theory of symmetric groups and their connections to Combinatorics.

Detailed lecture notes, written in Notability, will be posted and will often contain references. The references will be posted on the class webpage as well.

5. Assignments

Five homeworks (10% each), one in-class midterm (20%), its date is going to be determined based on preferences by students, and a take home final (30%). I hope that homeworks will be smaller than what I used to give in 380 (to show some mercy on the grader(s), for example), but we'll see how this goes...

6. Prerequisites

1) A good working knowledge of Linear Algebra (which can be achieved, say, in Math 240):

- vector spaces, subspaces, direct sums of (sub)spaces, quotient spaces on occasion;
- linear maps, their kernels, images, the rank-nullity theorems, etc.
- linear operators, eigenvalues, diagonalizability;
- Hermitian forms and Hermitian scalar products.

Knowing Multilinear Algebra is a plus, but a self-contained intro to tensor products of vector spaces will be given.

2) A good working knowledge of basics of Abstract algebra (which can be achieved, say, in Math 350):

- Groups, subgroups, homomorphisms, normal subgroups, quotient groups, isomorphism theorems, group actions on sets, cosets.
- More on groups, such as the center, the classification of finitely generated abelian groups.
- Rings and fields with examples. Ideals in rings, homomorphisms, etc.

We will need a few things about algebraic integers, but there will be a self-contained exposition. We will also need some versions of the fundamental theorem about symmetric polynomials, but this will also be explained (perhaps, only with a sketch of proof).

3) Bonus lectures will have their own prerequisites. You could also expect to encounter side remarks in the main lectures referring to things in Analysis, Commutative algebra, etc. These are not supposed to be crucial for understanding the lectures.