MATH 3800/5000, HOMEWORK 1, DUE SEPT 16

There are 5 problems worth 33 points total. Your score for this homework is the minimum of the sum of the points you've got and 28. Note that if the problem has several related parts, such as Problem 1, you can use previous parts to prove subsequent ones and get the corresponding credit. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 7 pts total. *This problem describes another construction of rings, the completion of a ring with respect to an ideal. Then we consider the special case: the ring of formal power series.*

Recall that in class we defined products of two rings $A_1 \times A_2$. This construction generalizes to an arbitrary collection of rings, even infinite: if X is an indexing set, and A_x, for $x \in X$, are commutative rings, then we can define the ring operations of $\prod_{x \in X} A_x$ component-wise, e.g. $(a_x)_{x \in X} + (b_x)_{x \in X} := (a_x + b_x)_{x \in X}$.

Let A be a ring and I be its ideal.

a, 1pt) Consider the following subset $\hat{A} \subset \prod_{i=1}^{\infty} A/I^i$: a tuple $(a_i)_{i \geqslant 1}$ with $a_i \in A/I^i$ lies in $\hat{A}$ if and only if for all $j > i$, the element a_i is the image of a_j under the projection $A/I^j \twoheadrightarrow A/I^i, a + I^j \mapsto a + I^i$. Show that

(1) $\hat{A}$ is a subring of $\prod_{i=1}^{\infty} A/I^i$,
(2) $\iota : A \to \hat{A}$ given by $a \mapsto (a + I^i)_{i>0}$ is a ring homomorphism,
(3) and $\hat{I}_j := \{(a_i)_{i>0} | a_k = 0, \forall k \leqslant j\}$ is an ideal in $\hat{A}$.

b, 3pts) Suppose that I is generated by elements $f_1, \ldots, f_n$. Show that

(1) $\hat{I}_1$ is generated by elements $\iota(f_1), \ldots, \iota(f_n)$,
(2) and $\hat{I}_j = \hat{I}_1^j$ for all $j > 1$.

An important example of a completion: when $A = \mathbb{Z}$ and $I = (p)$, where p is prime, the ring $\hat{A}$ is the ring of p-adic integers, it plays an important role in Algebraic Number theory.

c, 1pt) *We will concentrate on another important example: the ring of formal power series.* Let $A = B[x]$, where B is another ring, and $I = (x)$. Show that an element of $\hat{A}$ can be uniquely represented by a "formal power series", a sum $\sum_{i=0}^{\infty} b_i x^i$, where $b_i \in B$ (unlike with polynomials, we do not require that the sum is finite). Write formulas for the sum and product of two formal power series $\sum_{i=0}^{\infty} a_i x^i$ and $\sum_{i=0}^{\infty} b_i x^i$ in $\hat{A}$. *For $B = \mathbb{R}$ or $\mathbb{C}$, power series should be familiar from Calculus or Real/Complex Analysis. Unlike there, we do not require our power series to converge anywhere – which is why they are called formal.* The common notation for the ring of formal power series $\hat{A}$ is $B[[x]]$.

d, 1pt) *The ring of formal power series is closely related to the ring of polynomials. But it behaves differently, in fact, in many respects, it is simpler. The same applies to the p-adic integers vs the integers. In this part we discuss invertible elements in $B[[x]]$.* Prove that $\sum_{i=0}^{\infty} b_i x^i$ is invertible in $B[[x]]$ if and only if b_0 is invertible in B.

e, 1pt) Let B be a field. Show that the nonzero ideals in $B[[x]]$ are exactly (x^n) for $n \in \mathbb{Z}_{\geqslant 0}$.

f, 1pt) *And here is a similarity with the ring of polynomials.* Show that if B is a domain, then $B[[x]]$ is a domain.

Why the name "completion"? This is a special case of the completion of a topological (abelian) group. A related procedure is used to get $\mathbb{R}$ from $\mathbb{Q}$. Namely, we have a topology on A, where the ideals I^j, by definition, form a base of neighborhoods of zero. For example, we can define the limit of a sequence (a_i) to be $a \in A$ if for all $j > 0$ there is $n > 0$ with $a_i - a \in I^j$ for all $i > n$. We can define the notion of a Cauchy sequence in A in a similar fashion. There is the usual equivalence relation on the set of Cauchy sequences. The ring $\hat{A}$ is identified with the set of equivalence classes.

Problem 2, 4pts total. Let $\mathbb{F}$ be a field. Consider the ring A whose elements are formal sums of the form $\sum_{i=1}^n a_i x^{\lambda_i}$, where $a_1, \ldots, a_n \in \mathbb{F}$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ are nonnegative rational numbers, with natural addition and multiplication, i.e., $x^\lambda x^\mu := x^{\lambda+\mu}$ (informally, A is like the ring of polynomials but we allow fractional powers). Prove that:

a, 2pts) Every finitely generated ideal in A is principal. *Hint: use that every ideal in the ring of usual polynomials is principal.*

b, 2pts) The ideal of all elements, where the coefficient of x^0 is zero, is not finitely generated.

Problem 3, 5pts total. Consider the ideal $I = (2) \subset A = \mathbb{Z}[\sqrt{-5}]$.

a, 2pts) Prove that A/I is isomorphic to $\mathbb{F}_2[x]/(x^2)$, where $\mathbb{F}_2$ is the field with two elements.

b, 2pts) Find two elements of A that generate $\sqrt{I}$.

c, 1pt) Show that $\sqrt{I}$ is not principal.

The next problem ultimately provides an important tool to compute Hom modules in some way. It will be revisited much later in class.

Problem 4, 10pts total. Let L, M, N be A-modules. Then we have the composition map $\mathrm{Hom}_A(L, M) \times \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(L, N), (\varphi, \psi) \mapsto \psi \circ \varphi$.

a, 1pt) Prove that the composition map is A-bilinear.

Now consider four A-modules, M_1, M_2, M_3, N. Suppose we have A-linear maps $\varphi_1 : M_1 \to M_2, \varphi_2 : M_2 \to M_3$. Suppose that φ_2 is surjective, while $\operatorname{im} \varphi_1 = \ker \varphi_2$. Consider the maps, linear by part (a),

$$\tilde{\varphi}_1 : \mathrm{Hom}_A(M_2, N) \to \mathrm{Hom}_A(M_1, N), \psi_1 \mapsto \psi_1 \circ \varphi_1,$$
$$\tilde{\varphi}_2 : \mathrm{Hom}_A(M_3, N) \to \mathrm{Hom}_A(M_2, N), \psi_2 \mapsto \psi_2 \circ \varphi_2.$$

b, 3pts) Prove that $\tilde{\varphi}_2$ is injective and $\operatorname{im} \tilde{\varphi}_2 = \ker \tilde{\varphi}_1$.

c, 3pts) Suppose that, in the previous notation, $M_1 = A^{\oplus k}, M_2 = A^{\oplus \ell}$. So the map φ_1 is the multiplication by a matrix, denote it by $T = (t_{ji}), j = 1, \ldots, \ell, i = 1, \ldots, k$. Construct an isomorphism of A-modules between $\mathrm{Hom}_A(M_3, N)$ and the submodule of $N^{\oplus \ell}$ consisting of all ℓ-tuples $(n_1, \ldots, n_\ell)$ such that $\sum_{j=1}^{\ell} t_{ji} n_j = 0$ for all $i = 1, \ldots, k$.

d, 3pts) Let $A = \mathbb{Z}[x]$ and I be the ideal $(2, x)$ in A. Prove that for any homomorphism $\psi : I \to A$ there is $f \in A$ such that $\psi(b) = fb$ for all $b \in I$ and deduce an isomorphism $\mathrm{Hom}_A(I, A) = A$. *Hint: use d) for suitable* k, ℓ *and* φ_1 *as well as the fact that* A *is an UFD.*

A comment: For a finite dimensional vector space V *over a field* $\mathbb{F}$ *we can define its dual* $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$*. An important property of this construction is that* V^{**} *is naturally identified with* V*. Now for a module* M *over a ring* A *one could define its dual* M^* *as* $\mathrm{Hom}_A(M, A)$*. Part d) shows that for finitely generated modules* M *an isomorphism* $M^{**} \cong M$ *may fail.*

Problem 5, 7pts total. Let A be a commutative ring and $I \subset A$ be an ideal.

a, 2pts) Let $M_1, \ldots, M_k$ be A-modules. Prove that

$$I(M_1 \oplus M_2 \oplus \ldots \oplus M_k) = IM_1 \oplus IM_2 \oplus \ldots \oplus IM_k,$$

an equality of submodules in $M_1 \oplus M_2 \oplus \ldots M_k$. Use this to establish an isomorphism of A-modules

$$(M_1 \oplus M_2 \oplus \ldots M_k)/I(M_1 \oplus M_2 \oplus \ldots \oplus M_k) \cong M_1/IM_1 \oplus M_2/IM_2 \oplus \ldots M_k/IM_k.$$

b, 3pts) Use the previous part, the existence of a maximal ideal in A, and a result from Linear algebra to show that if $A^{\oplus k} \cong A^{\oplus \ell}$ as A-modules for nonnegative integers k, ℓ, then $k = \ell$.

c, 2pts) Generalize a) to arbitrary (infinite) direct sums and show that if $A^{\oplus X} \cong A^{\oplus Y}$ for sets X, Y, then X and Y have the same cardinality.