MATH 3800/5000, HOMEWORK 2, DUE OCT 7

There are 7 problems worth 35 points total. Your score for this homework is the minimum of the sum of the points you've got and 28. Note that if the problem has several related parts, you can use statements of the previous parts to prove subsequent ones and get the corresponding credit. You can also use the statements of problems in HW1 unless indicated otherwise. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

The first problem is about factorization into irreducibles in Noetherian domains. Its purpose is to illustrate what being Noetherian tells us about the structure of a ring. It also shows that not all of the three equivalent conditions of being Noetherian are created equal – from the point of view of a particular problem – I don't know solutions that would use finite generation of ideals, the AC termination condition gives a long-ish solution, while the most elegant solution is produced by using that every non-empty set of ideals has a maximal element with respect to inclusion.

Problem 1, 4pts. Let A be a Noetherian domain. Prove that every nonzero element of A decomposes into the product of irreducible elements and an invertible element.

Problem 2, 5pts. Let A be a Noetherian ring. Show that the formal power series ring $A[[x]]$ is Noetherian as well. *This is a more difficult problem. Hint: revisit the proof of the Basis theorem as well as a solution of part b) of Problem 1 in HW1.*

In fact, the completion of any Noetherian ring with respect to any ideal is still Noetherian.

Problem 3, 4pts total. We view the ring of Laurent polynomials $\mathbb{C}[x^{\pm 1}]$ as a $\mathbb{C}[x]$-module via the inclusion $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x^{\pm 1}]$. Consider the quotient $\mathbb{C}[x]$-module $M := \mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$.

1, 2pts) Describe all possible submodules of M.

2, 1pt) Prove that M is an Artinian $\mathbb{C}[x]$-module.

3, 1pt) Prove that M is not Noetherian as a $\mathbb{C}[x]$-module.

Problem 4, 4pts. Let A be a PID that is not a field and M be a finitely generated A-module. So we have $M \cong A^{\oplus k} \oplus \bigoplus_{i=1}^{\ell} A/(p_i^{d_i})$. Prove that the following two conditions are equivalent:

a) $k = 0$.

b) M is an Artinian A-module.

Problem 5, 5pts total. Consider the rings $A := \mathbb{C}[x,y]/(xy)$ and $B := \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}]$. Observe (not for credit) that we have ring homomorphisms $\pi_x : A \to \mathbb{C}[x]$ and $\pi_y : A \to \mathbb{C}[y]$ given by $f(x,y) + (xy) \mapsto f(x,0)$ and $f(x,y) + (xy) \mapsto f(0,y)$ for $f \in \mathbb{C}[x,y]$, respectively. Let $\varphi : A \to B$ be the homomorphism given by $a \mapsto (\pi_x(a), \pi_y(a))$ (where we view $\mathbb{C}[x], \mathbb{C}[y]$ as subrings of $\mathbb{C}[x^{\pm 1}], \mathbb{C}[y^{\pm 1}]$, respectively). Consider the element $\alpha := x + y + (xy) \in A$. Let $\iota : A \to A[\alpha^{-1}]$ denote the natural homomorphism.

a, 2pts) Show that there is a unique homomorphism $\tilde{\varphi} : A[\alpha^{-1}] \to B$ such that $\varphi = \tilde{\varphi} \circ \iota$.

b, 3pts) Show that $\tilde{\varphi}$ is an isomorphism.

Problem 6, 6pts total. Let A be a commutative ring. Let $\mathfrak{p}$ a prime ideal in A. *Recall that we write $?_{\mathfrak{p}}$ for the localization $?[(A \setminus \mathfrak{p})^{-1}]$.*

a, 3pts) Show that the map $\mathfrak{q} \mapsto \mathfrak{q}_{\mathfrak{p}}$ defines a bijection between

- the set of prime ideals of A contained in $\mathfrak{p}$,
- and the set of all prime ideals in $A_{\mathfrak{p}}$.

b, 3pts) Deduce that $A_{\mathfrak{p}}$ has a unique maximal ideal, and this ideal is $\mathfrak{p}_{\mathfrak{p}}$.

Commutative rings with only one maximal ideal are called "local" and are very important in the theory.

Problem 7, 7pts total. *Hom modules vs localizations, and how this helps to compute the modules of homomorphisms.* Let M, N be A-modules and S be a multiplicative subset in A.

1, 1pt) Prove that the map $\psi \mapsto \psi[S^{-1}] : \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}], N[S^{-1}])$ is A-linear.

The following two parts are harder.

2, 2pts) Suppose that M is finitely presented meaning that there are k, ℓ and an A-linear map $\varphi : A^{\oplus k} \to A^{\oplus \ell}$ such that $M \cong A^{\oplus \ell}/\operatorname{im}\varphi$. Prove that the A-linear map from part 1) factors into the composition of the natural homomorphism $\iota_{\mathrm{Hom}_A(M,N)} : \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,N)[S^{-1}]$ and an isomorphism $\mathrm{Hom}_A(M,N)[S^{-1}] \xrightarrow{\sim} \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}], N[S^{-1}])$.

Hint: something from HW1 should help...

3, 2pts) Now suppose that A is a Noetherian domain and $I \subset A$ is an ideal. Use the previous two parts to produce an A-module embedding of $\mathrm{Hom}_A(I, A)$ into $\mathrm{Frac}(A)$ whose image is $\{\alpha \in \mathrm{Frac}(A) | \alpha I \subset A\}$.

4, 2pts) Use part 3) to reprove a claim of Problem 8 in Homework 1: for the ideal $I = (2, x) \subset A = \mathbb{Z}[x]$, we have an isomorphism of A-modules $\mathrm{Hom}_A(I, A) \cong A$ (do not refer to your HW1 solution).