MATH 3800/5000, HOMEWORK 3, DUE OCT 28

There are 6 problems worth 33 points total. Your score for this homework is the minimum of the sum of the points you've got and 28. Note that if the problem has several related parts, you can use statements of the previous parts to prove subsequent ones and get the corresponding credit. You can also use the statements of problems in HW1 and HW2. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 6pts total. *The problem concerns rings of invariants and their properties.*

Let A be a domain and Γ be a subgroup in the group of ring automorphisms of A *(recall that an automorphism is an isomorphism $A \to A$; they do form a group).* Set $A^\Gamma := \{a \in A | \gamma a = a, \forall \gamma \in \Gamma\}$.

1, 1pt) Prove that A^Γ is a subring of A.

2, 1pt) Suppose A is normal. Prove that A^Γ is normal.

3, 2pts) Suppose Γ is finite. Prove that A is integral over A^Γ.

4, 2pts) Suppose B is a normal domain, $K := \mathrm{Frac}(B)$, L is a finite Galois extension of K with Galois group Γ, and A is the integral closure of B in L. Prove that $B = A^\Gamma$, the equality of subrings in L.

Problem 2, 4pts total. *This problem concerns localizations of normal and Dedekind domains.* Let A be a domain and $S \subset A$ be a multiplicative subset.

1, 2pts) Show that if A is normal, then $A[S^{-1}]$ is normal.

2, 2pts) Show that if A is Dedekind, then $A[S^{-1}]$ is Dedekind.

Problem 3, 5pts total. *This problem concerns various properties of Dedekind domains.* Let A be a Dedekind domain, and $\mathfrak{m} \subset A$ be a maximal ideal. In particular, $A/\mathfrak{m}$ is a field, and $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a vector space over this field.

1, 2pts) Show that $\dim_{A/\mathfrak{m}} \mathfrak{m}^k/\mathfrak{m}^{k+1} = 1$ for all k.

2, 1pt) Now assume A is local (which means that $\mathfrak{m}$ is the only maximal ideal). Prove that every nonzero ideal in A is of the form $\mathfrak{m}^k$ for some k.

3, 2pts) We continue to assume that A is local. Prove that A is a PID.

Problem 4, 6pts. *Here we consider rings of algebro-geometric nature.* Let $\mathbb{F}$ be an algebraically closed field, and f an irreducible polynomial in $\mathbb{F}[x, y]$ so that $A := \mathbb{F}[x, y]/(f)$ is a domain.

1, 2pts) Prove that every nonzero prime ideal in A is maximal (*hint: use the Noether normalization lemma – and some other tools*).

2, 2pts) Prove that if A is Dedekind, then the gradient $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is nonzero at any point of the zero locus $\{(x, y)|f(x, y) = 0\}$.

In other words, for A to be Dedekind, the algebraic curve $\{(x,y)|f(x,y)=0\}$ must be smooth. The converse is true as well: it turns that for a Noetherian domain such that every nonzero prime ideal is maximal, being integrally closed is equivalent to the condition of 1) of Problem 3 for $k=1$. At this point of the class we are missing some ingredients to prove the statement, most notably the Nakayama lemma.

3, 2pts) *And here's a concrete example of failure of normality.* Show that $A := \mathbb{F}[x,y]/(y^2-x^3)$ is not normal (*you may use without a proof that the polynomial y^2-x^3 is irreducible*). Furthermore, prove that the integral closure of A in $\operatorname{Frac}(A)$ is $\mathbb{F}[y/x]$.

Problem 5, 6pts total. *A general principle is that, basically, every construction in Commutative algebra has an incarnation in Algebraic geometry. Here we investigate this principle for algebra homomorphisms.* Let $X \subset \mathbb{F}^\ell, Y \subset \mathbb{F}^m, Z \subset \mathbb{F}^n$ be algebraic subsets, and $A := \mathbb{F}[X], B := \mathbb{F}[Y], C := \mathbb{F}[Z]$ be their algebras of polynomial functions. A map $\varphi : X \to Y$ is called <u>polynomial</u> if there are elements $f_1, \ldots, f_m \in \mathbb{F}[X]$ such that $\varphi(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in X$ (in particular, $(f_1(x), \ldots, f_m(x)) \in Y$ for all $x \in X$).

1, 2pts) Prove that the map $g \mapsto g \circ \varphi$ is an algebra homomorphism $\operatorname{Fun}(Y, \mathbb{F}) \to \operatorname{Fun}(X, \mathbb{F})$ that restricts to $\mathbb{F}[Y] \to \mathbb{F}[X]$. Denote this restriction by φ^*.

2, 3pts) Prove that for every algebra homomorphism $\zeta : \mathbb{F}[Y] \to \mathbb{F}[X]$, there is a unique polynomial map $\varphi : X \to Y$ with $\zeta = \varphi^*$. *So, we have a natural bijection between the polynomial maps $X \to Y$ and algebra homomorphism $\mathbb{F}[Y] \to \mathbb{F}[X]$.*

3, 1pt) Prove that $\mathrm{id}_X^* = \mathrm{id}_{\mathbb{F}[X]}$ and for any two polynomial maps $\varphi : X \to Y, \psi : Y \to Z$, we have $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Problem 6, 6pts. Let $\mathbb{F}$ be an algebraically closed field. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ and I be an ideal in A. Set $J = \sqrt{I}$.

1, 2pts) Prove that there is $n > 0$ such that $J^n \subset I$.

2, 4pts) Use part (1) (as well as additional properties) to conclude that the following three claims are equivalent:

(a) $V(I) \subset \mathbb{F}^n$ is finite.
(b) J has finite codimension in A, i.e., $\dim_{\mathbb{F}} A/J < \infty$.
(c) I has finite codimension in A.