MATH 3800/5000, HOMEWORK 4, DUE NOV 16

There are 5 problems worth 32 points total. Your score for this homework is the minimum of the sum of the points you've got and 28. Note that if the problem has several related parts, you can use statements of the previous parts to prove subsequent ones and get the corresponding credit. You can also use the statements of problems in HW1 – HW3. The text in italic below is meant to to be comments to a problem but not a part of it.

Problem 1, 6pts. Let A be a commutative ring.

a, 3pts) Show that the object $A^{\oplus k}$ in A-Mod represents the functor $\text{For}^k : A\text{-Mod} \to \text{Sets}$, the product of k copies of the forgetful functor For. *Here you are responsible for verifying axioms of functor morphisms.*

b, 3pts) Show that the monoid of endomorphisms of For is isomorphic to A viewed as a monoid with respect to multiplication.

Problem 2, 5pts. Let FinGroups denote the category of finite groups. Show that the coproduct $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ doesn't exist in FinGroups. *Hint: the diherdral group D_n, i.e., the group of symmetries of the regular n-gon, has $2n$ elements and is generated by two elements of order 2 – you are welcome to use these facts in your solution without proof. And, in general, two non-trivial finite groups do not have the coproduct in* FinGroups *– but have it in* Groups.

Problem 3, 4pts. *Compositions of adjoint functors.*

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}, F' : \mathcal{D} \to \mathcal{E}, G' : \mathcal{E} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that F is left adjoint to G and F' is left adjoint to G'. Prove that $F'F$ is left adjoint to GG'. *In this problem you are responsible for checking the commutativity of diagrams in the definition of adjoint functors.*

Problem 4, 7pts total. Let $A := \mathbb{C}[x, y]$ and consider the ideal $I := (x, y)$.

a, 4pts) Prove an isomorphism

$$I \otimes_A I \cong A^{\oplus 4} / \operatorname{Span}_A((y, -x, 0, 0), (0, 0, y, -x), (y, 0, -x, 0), (0, y, 0, -x)).$$

b, 3pts) Show that the natural homomorphism

$$I \otimes_A I \to I \otimes_A A = I, a \otimes b \mapsto ab,$$

is not injective.

Problem 5, 10pts total. *This problem introduces important constructions with modules: their tensor and symmetric algebras.* Let A be a commutative ring and M be an A-module. Let $M^{\otimes i}$ denote the i-fold tensor product $M \otimes_A M \otimes_A \ldots \otimes_A M$ (with $M^{\otimes 0} = A, M^{\otimes 1} = M$).

Consider the A-module $T_A(M) := \bigoplus_{i=0}^{\infty} M^{\otimes i}$. We define an associative A-algebra structure on $T_A(M)$ as follows: for $u \in M^{\otimes i}, v \in M^{\otimes j}$ their product is $u \otimes v$ in $M^{\otimes i} \otimes_A M^{\otimes j}$, which, as we know, is identified with $M^{\otimes (i+j)}$. Then the product is extended to the entire A-module $T_A(M)$ by linearity. This equips $T_A(M)$ with the structure of an associative A-algebra with unit $1 \in A$. *Check this, not for credit.*

1, 2pts) Let $\varphi : M \to N$ be an A-module homomorphism. Produce a natural A-algebra homomorphism $T_A(\varphi) : T_A(M) \to T_A(N)$. Show that T_A is a functor A-Mod $\to$ A-Alg. *Here you are responsible for both construction of a homomorphism and checking the functor axioms.*

2, 2pts) Show that the functor T_A is left adjoint to the forgetful functor A-Alg $\to$ A-Mod. *Here and in 4), you are only responsible for constructing a natural isomorphism between the two Hom sets but not for checking the commutativity of two diagrams in the definition of adjoint functors.*

3, 2pts) Define the *symmetric algebra*, $S_A(M)$ as the quotient $T_A(M)$ by the A-linear span of the elements of the form $a \otimes (m_1 \otimes m_2 - m_2 \otimes m_1) \otimes b$ for $a, b \in T_A(M), m_1, m_2 \in M$. Show that $S_A(M)$ is a commutative A-algebra and exhibit S_A is a functor A-Mod $\to$ A-CommAlg. *Here you are only responsible for constructing a natural algebra homomorphism.*

4, 2pts) Show that $S_A : A\text{-Mod} \to A\text{-CommAlg}$ is left adjoint to the forgetful functor A-CommAlg $\to$ A-Mod.

5, 2pts) Let $M := A^{\oplus k}$. Let For denote the forgetful functor A-CommAlg $\to$ Sets. Show that $S_A(M)$ represents the functor $\text{For}^{\times k} : A\text{-CommAlg} \to \text{Sets}$ and use this to deduce that $S_A(M) \cong A[x_1, \ldots, x_k]$ as A-algebras. *So, the symmetric algebra may be viewed as a generalization of the polynomial algebra.*