MATH 3800/5000, HOMEWORK 5, DUE DEC 4

There are 5 problems worth 32 points total. Your score for this homework is the minimum of the sum of the points you've got and 28. Note that if the problem has several related parts, you can use statements of the previous parts to prove subsequent ones and get the corresponding credit. You can also use the statements of problems in HW1 – HW4. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 6pts total. *Some more tensor-Hom adjunction.*

Let $\varphi : A \to B$ be a ring homomorphism, and L be a B-module.

1, 2pts) For an A-module N, show that there is a unique B-module structure on $\operatorname{Hom}_A(L, N)$ such that, first, $[b\varphi](\ell) = \varphi(b\ell)$ for all $b \in B, \varphi \in \operatorname{Hom}_A(L, N), \ell \in L$ and, second, the A-module structure obtained by pullback coincides with the original A-module structure.

2, 1pt) Present $\operatorname{Hom}_A(L, ?)$ as a functor $A\text{-Mod} \to B\text{-Mod}$. *The notation for the resulting functor is* $\underline{\operatorname{Hom}}_A(L, ?)$.

3, 2pts) Show that the functor $\varphi^*(L \otimes_B ?) : B\text{-Mod} \to A\text{-Mod}$ is left adjoint to the functor in 2). *You need to produce a natural bijection but don't need to check the commutative diagrams.*

4, 1pt) In particular, deduce that $\underline{\operatorname{Hom}}_A(B, ?)$ is right adjoint to φ^*.

Problem 2, 8pts total. *This problem studies products of algebraic subsets and their interactions with tensor products of algebras.* Let $\mathbb{F}$ be an algebraically closed field, and $X \subset \mathbb{F}^n, Y \subset \mathbb{F}^m$ be algebraic subsets.

1, 2pt) Show that $X \times Y \subset \mathbb{F}^{n+m}$ is an algebraic subset.

2, 3pts) Show that $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y]$ has no nonzero nilpotent elements. *Hint: apply homomorphisms* $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] \to \mathbb{F}[Y]$ *evaluating at points of* X.

3, 3pts) Establish a natural isomorphism $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] \xrightarrow{\sim} \mathbb{F}[X \times Y]$.

The conclusion is that the tensor product of algebras of functions corresponds to the product of algebraic subsets. This may serve as a motivation to consider tensor products of algebras.

Problem 3, 6pts total. *Exactness and Homs.* Let M_1, M_2, M_3 be A-modules and let $\varphi_1 : M_1 \to M_2$ and $\varphi_2 : M_2 \to M_3$ be A-linear maps with $\varphi_2\varphi_1 = 0$.

1, 2pts) Prove that the sequence $0 \to M_1 \to M_2 \to M_3$ is exact if and only if the corresponding sequence

$$0 \to \operatorname{Hom}_A(N, M_1) \to \operatorname{Hom}_A(N, M_2) \to \operatorname{Hom}_A(N, M_3)$$

is exact for every A-module N.

2, 4pts) Prove that the sequence $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if the corresponding sequence

$$0 \to \operatorname{Hom}_A(M_3, N) \to \operatorname{Hom}_A(M_2, N) \to \operatorname{Hom}_A(M_1, N)$$

is exact for every A-module N.

Problem 4, 7pts total. *Adjunction implies opposite exactness.* Let A, B be rings. Let an additive functor $F : A\text{-Mod} \to B\text{-Mod}$ be left adjoint to an additive functor $G : B\text{-Mod} \to A\text{-Mod}$.

1, 3pts) Show that F is right exact and G is left exact.

2, 4pts) Show that the following claims are equivalent:

(i) F sends projective modules to projective modules.
(ii) $F(A)$ is a projective B-module.
(iii) G is exact.

This problem illustrates an important principle: we can derive favorable properties of functors just by knowing that they are left or right adjoint.

Problem 5, 5pts. *And the ideal (x, y) arises as a counterexample again!* Show that the ideal $(x, y) \subset \mathbb{C}[x, y]$ is not projective as a module over $\mathbb{C}[x, y]$.