Lecture 10: Localization III / Integral extensions I

1) Localization of modules, cont'd.

2) Finite & integral algebras.

Ref: [AM], Sections 3, 3.1, 5.1

1) Localization of modules cont'd.

1.0) Reminder

Let $S\subset A$ be a multiplicative subset in a commutative ring, so that we can form the localization $A[S^{-1}]$. Let M be an A-module. We form the $A[S^{-1}]$-module

$$M[S^{-1}]=\{\frac{m}{s}\,|\, m\in M, s\in S\},$$

It comes w. an A-linear map $\iota_M: M\to M[S^{-1}], m\mapsto \frac{m}{1}$.

The pair $(M[S^{-1}],\iota_M)$ has the following universal property: for an $A[S^{-1}]$-module N & A-linear map $\xi: M\to N$ $\exists!$ $A[S^{-1}]$-linear $\xi': M[S^{-1}]\to N$ w. $\xi'=\xi\circ\iota_M$, it's given by $\xi(\frac{m}{s})=\frac{1}{s}\xi(m)$.

In particular, to $\varphi\in Hom_A(M_1,M_2)$ we can assign $\varphi[S^{-1}]\in Hom_{A[S^{-1}]}(M_1[S^{-1}],M_2[S^{-1}])$ by $\varphi[S^{-1}](\frac{m}{s})=\frac{\varphi(m)}{s}$.

See Sec 2.2 in Lec 9 for details.

In this section we'll study interaction of localization w. kernels, images, quotients & direct sums, which will give us some tools to compute (in some sense) localizations of modules.

1.1) Localization vs kernels and images.

Our next task is to relate $\ker\varphi[S^{-1}]$, $\operatorname{im}\varphi[S^{-1}]$ to $\ker\varphi$, $\operatorname{im}\varphi$.

Proposition: Let M, N be A-modules & $\varphi \in Hom_A(M,N)$

i) $\ker(\varphi[S^{-1}]) = (\ker\varphi)[S^{-1}]$

ii) $im(\varphi[S^{-1}]) = (im\,\varphi)[S^{-1}]$

Proof: i) First, we check $\ker(\varphi[S^{-1}]) \subset (\ker\varphi)[S^{-1}]$

$\ker(\varphi[S^{-1}]) = \{\frac{m}{s} \in M[S^{-1}] \mid 0 = \varphi[S^{-1}](\frac{m}{s}) = \frac{\varphi(m)}{s} \iff \exists$ $u \in S \mid u\varphi(m) = 0 \iff um \in \ker\varphi\} \subseteq [\frac{um}{us} = \frac{m}{s}] \subseteq (\ker\varphi)[S^{-1}]$.
Now $(\ker\varphi)[S^{-1}] = \{\frac{m}{s} \mid \varphi(m) = 0\} \subset \ker(\varphi[S^{-1}])$, finishing (i).

(ii) $im(\varphi[S^{-1}]) = \{\varphi[S^{-1}](\frac{m}{s}) = \frac{\varphi(m)}{s}\} = (im\,\varphi)[S^{-1}]$. $\square$

Corollary: Let M be A-module, $M' \subset M$ be an A-submodule.
Then there's a natural $A[S^{-1}]$-module isomorphism
$(M/M')[S^{-1}] \xrightarrow{\sim} M[S^{-1}]/M'[S^{-1}]$.

Proof:
Apply Proposition to $\varphi: M \twoheadrightarrow M/M'$, $m \mapsto m + M'$. Then
$im(\varphi[S^{-1}]) = (im\,\varphi)[S^{-1}] = (M/M')[S^{-1}]$; $\ker(\varphi[S^{-1}]) =$
$(\ker\varphi)[S^{-1}] = M'[S^{-1}] \Rightarrow M[S^{-1}]/M'[S^{-1}] \xrightarrow{\sim} (M/M')[S^{-1}]$ $\square$

1.2) Localizations vs direct sum.

Let I be a set and M_i, $i \in I$, be A-modules so that we can form the direct sum $\bigoplus_{i \in I} M_i$.

Lemma: There's a natural isomorphism $\bigoplus_{i\in I}(M_i[S^{-1}]) \xrightarrow{\sim} (\bigoplus_{i\in I} M_i)[S^{-1}]$.

Proof:

Set $M = \bigoplus_{i\in I} M_i$. Consider the map $\zeta: M \to \bigoplus_{i\in I}(M_i[S^{-1}])$, $(m_i) \mapsto (\frac{m_i}{1})$, it's A-linear. By the universal property, it lifts to the $A[S^{-1}]$-linear map $\zeta': M[S^{-1}] \to \bigoplus_{i\in I}(M_i[S^{-1}])$, $\frac{(m_i)}{s} \mapsto (\frac{m_i}{s})$.

• ζ' is injective ($\zeta'(\frac{(m_i)}{s}) = 0 \Leftrightarrow \frac{m_i}{s} = 0 \ \forall i$) Let $I_0 = \{i \mid m_i \neq 0\}$. This is a finite subset of I. For $i \in I_0$, $\frac{m_i}{s} = 0 \Leftrightarrow \exists u_i \in S \mid u_i m_i = 0$. Take $u = \prod_{i\in I_0} u_i$ so that $um_i = 0 \ \forall i \in I \Rightarrow \frac{(m_i)}{s} = 0$

• ζ' is surjective ($\forall (\frac{m_i}{s_i}) \in \bigoplus_{i\in I}(M_i[S^{-1}]) \Rightarrow (\frac{m_i}{s_i}) \in im\,\zeta'$). Let $I_1 := \{i \in I \mid \frac{m_i}{s_i} \neq 0\}$ - finite set. Set $s := \prod_{i\in I_0} s_i$, $\tilde{m}_i = (\prod_{j\in I_0\setminus\{i\}} s_j) m_i$ so that $\frac{m_i}{s_i} = \frac{\tilde{m}_i}{s} \ \forall i \in I_1$. Set $\tilde{m}_i := 0$ for $i \notin \tilde{I}_1$. Then $\frac{(\tilde{m}_i)}{s} \mapsto (\frac{m_i}{s_i})$, showing the surjectivity. □

Example: $M = A^{\oplus I}$. So $M[S^{-1}] \simeq A[S^{-1}]^{\oplus I}$ - the localization of a free module is free.

1.3) A way to compute $M[S^{-1}]$

Assume M is finitely presented: $\exists A^{\oplus \ell} \overset{\pi}{\twoheadrightarrow} M$ with finitely generated kernel. Choosing generators in $\ker\pi$ gives a surjection $A^{\oplus k} \twoheadrightarrow \ker\pi$ hence an A-linear map $\varphi: A^{\oplus k} \to A^{\oplus \ell}$ w. $M \simeq A^{\oplus k}/im\,\varphi$, one can view this as a presentation of M by generators & relations.

By Sec 1.1, $M[S^{-1}] \simeq A^{\oplus \ell}[S^{-1}]/im\,\varphi[S^{-1}]$ & by example in Sec. 1.2

$A^{\oplus \ell}[S^{-1}] = A[S^{-1}]^{\oplus \ell}$. So $M[S^{-1}] \simeq A[S^{-1}]^{\oplus \ell}/\operatorname{im}\varphi[S^{-1}]$. So we need to compute $\varphi[S^{-1}]$. Recall (Sec 3.2 of Lec 2) that $\varphi: A^{\oplus k} \to A^{\oplus \ell}$ is given by a matrix $\Phi \in \operatorname{Mat}_{\ell \times k}(A)$, $\Phi = (a_{ij})$: if we view el'ts of $A^{\oplus k}, A^{\oplus \ell}$ as column vectors, then $\varphi(v) = \Phi v$. Then $\varphi[S^{-1}]: A[S^{-1}]^{\oplus k} \to A[S^{-1}]^{\oplus \ell}$ is given by the matrix $\left(\frac{a_{ij}}{1}\right)$ (exercise).

2) Finite and integral algebras.

In what follows A is a commutative ring & B is a commutative A-algebra (a ring w. fixed homomorphism from A)

The concepts of finite & integral A-algebras (and related results) generalize the concepts of finite & algebraic field extensions (and related results). They are important for Algebraic Number theory as we will see in subsequent lectures.

2.1) Main definitions.

Recall (Sec 2.2 of Lec 5) that B is finitely generated (as an A-algebra) if $\exists\, b_1, \dots b_n \in B$ (generators) s.t. $\forall b \in B\ \exists F \in A[x_1, \dots x_n] \mid b = F(b_1, \dots b_n)$.

Definition: We say that B is finite over A if it is a finitely generated A-module.

In particular, finite $\Rightarrow$ finitely generated but not vice versa: $A[x]$ is finitely generated as an A-algebra but is not finite.

Definition: Let B be a commutative A-algebra.

- $b \in B$ is **integral** over A if $\exists$ monic (i.e. leading coeff $= 1$) $f \in A[x] \mid f(b) = 0$.
- B is **integral** over A if $\forall\, b \in B$ is integral (over A).

Exercise: If B is integral over A & C is a quotient of B, then C is integral over A.

Rem: If $A \hookrightarrow B$ we can view A as a subring of B. We call B an extension of A and talk about finite/integral extensions.

2.2) Examples

1) Let $A := K$, $B := L$ be fields. Here any homomorphism from A is injective, so L is a field extension of K. "L is finite over K" is the usual notion from the study of field extensions. And L is integral over K iff L is algebraic over K: if $\ell \in L$ & $g \in K[x]$ are s.t. $g(\ell) = 0$ (i.e. ℓ is algebraic over K) & $g = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ w. $a_n \neq 0$, then set $f = a_n^{-1} g$, it's monic and satisfies $f(\ell) = 0$. So ℓ is integral.

2) Let $f(x) \in A[x]$ be a monic polynomial. Then $\bar{x} := x + (f) \in B := A[x]/(f)$ is tautologically integral$/A$. Also note that B is finite over A (spanned by $1, \bar{x}, \ldots \bar{x}^{d-1}$ for $d := \deg f$).
Below we'll see that B is integral over A.

3) Non-example: Let A be a domain & $g(x) = a_n x^n + \ldots + a_0$ where a_n is not invertible (but $a_n \neq 0$). Then $\bar{x} = x + (f) \in B' := A/(g)$ is not integral over A: if $h \in A[x]$ is a monic polynomial w. $h(\bar{x}) = 0$, then $h \vdots g$ in $A[x]$, which is impossible.

2.3) Finite vs integral

Reminder: for field extensions: finite $\Leftrightarrow$ [algebraic & finitely generated (as a field extension)]. This generalizes to rings.

Thm: Let B be an A-algebra. TFAE

(a) B is integral and finitely generated A-algebra.

(b) B is finite A-algebra.

The proof of (a) $\Rightarrow$ (b) is based on the following lemma. Note that if A_1 is an A-algebra & A_2 is an A_1-algebra, then A_2 is also an A-algebra: for the homomorphism $A \to A_2$ take the composition $A \to A_1 \to A_2$.

Lemma 1: Suppose A_1 is finite over A & A_2 is finite over A_1. Then A_2 is finite over A.

Proof: Have $a_1, \ldots, a_k \in A_1$ & $b_1, \ldots b_\ell \in A_2$ s.t. $A_1 = \mathrm{Span}_A(a_1, \ldots a_k)$, $A_2 = \mathrm{Span}_{A_1}(b_1, \ldots, b_\ell)$.

Exercise: $A_2 = \mathrm{Span}_A(a_j b_i \mid i = 1, \ldots, \ell,\ j = 1, \ldots k)$ $\square$

Notation: For an A-algebra B & $b_1, \dots b_k \in B$ we write $A[b_1, \dots b_k]$ for the A-subalgebra of B generated by $b_1, \dots b_k$.

Proof of (a) $\Rightarrow$ (b): say B is generated by some elements $b_1, \dots b_k$ as an A-algebra. We induct on k.

Base: $k=1$: B is generated by b as A-algebra. b is integral over A, let $f \in A[x]$ be monic s.t. $f(b)=0$. Then the unique A-algebra homomorphism $A[x] \to B$ w. $x \mapsto b$ factors as $A[x] \twoheadrightarrow A[x]/(f) \to B$. Since b generates B, have $A[x] \twoheadrightarrow B \Rightarrow A[x]/(f) \twoheadrightarrow B$. By Example 2 above, $A[x]/(f)$ is fin. gen'd A-module $\Rightarrow B$ is fin. gen'd A-module.

Step 2: B is generated by $b_2, \dots b_k$ ($k-1$ el'ts) over $\tilde{A} := A[b_1]$. By inductive assumption, B is finite over $\tilde{A}$. Now we apply Lemma 1 (to $A_1 = \tilde{A}$, $A_2 = B$) to finish the proof. $\square$

(b) $\Rightarrow$ (a) will be proved in the next lecture.