Lecture 15: Connections to Algebraic geometry, II.

1) Prime ideals, irreducibility & components.
2) Algebra of polynomial functions
3) Geometric significance of localization.

Refs: [V], Sec 9.6; [E], Intro to Sec 2, Sec 3.8.

0) Reminder from Lec 14.

Below $\mathbb{F}$ denotes an algebraically closed field.

Here are some results & definitions from Lec 14. For a subset $X\subset \mathbb{F}^n$, we write $I(X)$ for $\{f\in \mathbb{F}[x_1,\dots x_n] \mid f(\alpha)=0\ \forall \alpha\in \mathbb{F}^n\}$.

For ideal $I\subset \mathbb{F}[x_1,\dots,x_n]$ we write $V(I)$ for $\{\alpha\in\mathbb{F}^n \mid f(\alpha)=0\ \forall f\in I\}$. We know that $V(I)=V(\sqrt{I})$ (iii) in Sec 1.4).

I) Corollary in Sec 1.2: $\forall$ ideal $I\subset \mathbb{F}[x_1,\dots x_k]$, there's a bijection between {maximal ideals in $\mathbb{F}[x_1,\dots x_n]/I$} & $V(I)$, it sends $\alpha\in V(I)$ to $\mathfrak{m}_\alpha=\{f+I \mid f(\alpha)=0\}$

II) Proposition in Sec 1.4: $I\mapsto V(I)$ & $X\mapsto I(X)$ are mutually inverse bijections between {radical ideals (i.e. $I=\sqrt{I}$) in $\mathbb{F}[x_1,\dots x_n]$} and {algebraic subsets in $\mathbb{F}^n$}. Moreover (Exercise in Sec 1.4), these maps reverse inclusions.

III) Lemma in Sec 1.4: For ideals $I,J\subset \mathbb{F}[x_1,\dots x_n] \Rightarrow V(IJ)=V(I\cap J)$ $=V(I)\cup V(J)$.

Remark: Here's the (double) point of what's going to happen in this lecture (as well as in some future lectures & homeworks).

- Algebraic geometry studies the geometry of spaces defined by polynomial equations (of which algebraic subsets of $\mathbb{F}^n$ are basic examples). Most constructions/definitions/results in Algebraic geometry (ultimately) can be translated to the language of Commutative algebra.
- Most constructions in Commutative algebra have geometric interpretation/meaning.

Below we are going to see some examples of this

1) Prime ideals, irreducibility & components.

1.1) Prime ideals vs irreducible subsets

Let $\mathfrak{p} \subset \mathbb{F}[x_1,\dots x_n]$ be a prime ideal. It's radical: $a_1 a_2 \in \mathfrak{p} \Rightarrow a_1$ or $a_2 \in \mathfrak{p}$, so $a^n \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$. Our question is: what can we say about $V(\mathfrak{p})$?

Definition: an algebraic subset X in $\mathbb{F}^n$ is called

- irreducible: if X cannot be represented as $X_1 \cup X_2$, where $X_i \subsetneq X$ is algebraic.
- reducible, else.

Proposition: Let $I \subset \mathbb{F}[x_1,\dots x_n]$ be a radical ideal. TFAE

1) I is prime

2) $V(I)$ is irreducible.

Sketch of proof:

i) I is not prime $\Leftrightarrow \exists\, I_1, I_2 \supsetneq I$ w. $I_1 I_2 \subset I$

ii) $I_i \supsetneq I \Rightarrow \sqrt{I_i} \supsetneq I \Rightarrow V(I_i) = V(\sqrt{I_i}) \subsetneq V(I)$ by II) in Sec 0 (we have $V(\sqrt{I_i}) \neq V(I)$ b/c both $I, \sqrt{I_i}$ are radical). So $V(I_1) \cup V(I_2) \subset V(I)$.

iii)) $I_1 I_2 \subset I \Rightarrow V(I_1) \cup V(I_2) = V(I_1 I_2) \supset V(I) \Rightarrow V(I_1) \cup V(I_2) = V(I)$.

This shows 2) $\Rightarrow$ 1). We leave 1) $\Rightarrow$ 2) as an exercise. $\square$

Example: Let $f \in \mathbb{F}[x_1, \ldots x_n] \setminus \mathbb{F}$. Decompose $f = g_1^{d_1} \ldots g_e^{d_e}$, where $g_i \neq g_j$ are irreducible. Then $\sqrt{(f)} = (g_1 \ldots g_e)$ (cf. Example in Sec 1 of Lec 2) b/c $\mathbb{F}[x_1 \ldots x_n]$ is UFD; $V(f) = [\mathrm{III}] = \bigcup_{i=1}^{e} V(g_i)$. So $V(f)$ is irreducible $\Leftrightarrow$ $e=1$. For instance, if $n=2$ & $f = x_1^{d_1} x_2^{d_2}$ for $d_1, d_2 > 0$, then $V(f)$ is the union of the lines $x_1 = 0$ & $x_2 = 0$, reducible.

1.2) Irreducible components.

Theorem: Let X be an algebraic subset in $\mathbb{F}^n$. Then

a) $\exists$ irreducible algebraic subsets $X_1, \ldots X_k$ s.t. $X = \bigcup_{i=1}^{k} X_i$.

b) For $X_1, \ldots X_k$ we can take maximal (w.r.t. inclusion) irreducible algebraic subsets contained in X.

Note that (b) recovers $X_1, \ldots X_k$ uniquely.

Def'n: $X_1, \ldots X_k$ from b) are called the irreducible components of X.

Example: In the notation of the previous example, the irreducible components of $V(f)$ are $V(g_1),\dots, V(g_\ell)$.

Proof of Theorem:

a) Assume the contrary: $\exists\ X \neq$ finite union of irreducibles $\Leftrightarrow$ the set $\mathcal{A}$ of all such X's is $\neq \emptyset$. $\rightsquigarrow$ nonempty set $\{I(X) \mid X \in \mathcal{A}\}$. Since $\mathbb{F}[x_1,\dots x_n]$ is Noetherian, every nonempty set of ideals has maximal (w.r.t $\subset$) element. Pick $X' \in \mathcal{A}$ s.t. $I(X')$ is maximal in $\{I(X) \mid X \in \mathcal{A}\} \overset{[\mathrm{II})]}{\Leftarrow\Rightarrow} X'$ is minimal in $\mathcal{A}$ w.r.t. $\subset$. But X' is reducible b/c $X' \in \mathcal{A} \Leftrightarrow X' = X^1 \cup X^2$ w. $X^i \subsetneq X'$ $\Rightarrow [X'$ is min'l in $\mathcal{A}]$ $X^i \notin \mathcal{A} \rightsquigarrow X^i = \bigcup\limits_j X^i_j$ (finite unions of irreducibles) $\rightsquigarrow X' = \bigcup\limits_j X^1_j \cup \bigcup\limits_j X^2_j$ – contradicts $X' \in \mathcal{A}$.

b) $X = \bigcup\limits_{i=1}^{k} X_i$, we assume that none of X_i's is contained in another. Need to show: X_i is max'l irreducible (exercise) & if $Y \subset X$ max'l irreducible $\Rightarrow Y = X_i$ (for autom. unique i). To prove this, we observe $Y = \bigcup\limits_{i=1}^{k} (Y \cap X_i)$; since Y is irreducible $\Rightarrow Y = Y \cap X_i$ for some i $\Rightarrow Y \subset X_i$, but since Y is maximal, $Y = X_i$. □

Remark (alg. formulation of Thm): Let $I \subset \mathbb{F}[x_1,\dots x_n]$ be a radical ideal. Then $I = \bigcap\limits_{i=1}^{k} I_i$, where I_i is prime; and we can recover I_i's uniquely if we assume they are minimal (w.r.t $\subseteq$) w. $I \subset I_i$. To prove this is an exercise.

Remark: the same statement is true if $\mathbb{F}[x_1,\dots,x_n]$ is replaced w. arbitrary Noetherian ring. There's a suitable generalization to arbitrary ideals: primary decomposition, [AM], Ch. 4 & 7.1.

2) Algebra of polynomial functions.

In most geometric contexts, the spaces being studied come with a distinguished class of functions - that play an important role in studying the space. E.g. for C^∞-submanifolds in $\mathbb{R}^n$ (or abstract C^∞-manifolds one considers C^∞-functions). For algebraic subsets of $\mathbb{F}^n$ this role is played by polynomial functions.

Let X be an algebraic subset of $\mathbb{F}^n$ & $I = I(X)$. Consider the set $Fun(X,\mathbb{F})$ of all functions $X \to \mathbb{F}$. This is an $\mathbb{F}$-algebra w. point-wise operations, e.g. $(f_1 f_2)(\alpha) = f_1(\alpha) f_2(\alpha)$. It admits a homomorphism $\mathbb{F}[x_1,\dots x_n] \longrightarrow Fun(X,\mathbb{F})$, $f \mapsto f|_X$, with kernel I.

Definition: The algebra of polynomial functions, $\mathbb{F}[X]$ is the image of $\mathbb{F}[x_1,\dots x_n]$ in $Fun(X,\mathbb{F})$. Note that it's identified w. $\mathbb{F}[x_1,\dots x_n]/I$.

Exercise

1) $\mathbb{F}[X]$ has no nonzero nilpotent elements ($f \neq 0 \mid \exists\, \ell > 0$ w. $f^\ell = 0$).
2) There's a bijection between:
   - Radical ideals $J \subset \mathbb{F}[X]$

• algebraic subsets $Y \subset X$ (i.e. algebraic subsets in $\mathbb{F}^n$ contained in X).

It sends $Y \subset X$ to $\{f \in \mathbb{F}[X] \mid f|_Y = 0\}$ (hint: use II) in Sec 0).

3) $\{$max. ideals in $\mathbb{F}[X]\} \xleftrightarrow{\sim} X$: $\alpha \in X \mapsto \{f \in \mathbb{F}[X] \mid f(\alpha) = 0\}$ (hint: use I) in Sec 0)

Remark: We can recover $X \subset \mathbb{F}^n$ from $\mathbb{F}[X]$ & generators $\bar{x}_i := x_i + I$. Namely, $I = \{F \in \mathbb{F}[x_1, \dots x_n] \mid F(\bar{x}_1, \dots \bar{x}_n) = 0$ in $\mathbb{F}[X]\} \rightsquigarrow X = V(I)$

Example: Let $X = \{(x_1, x_2) \mid f(x_1, x_2) = 0\} \subset \mathbb{F}^2$ for irreducible $f \in \mathbb{F}[x_1, x_2]$ (f) is radical & equal to $I(X)$. Then $\mathbb{F}[X] = \mathbb{F}[x_1, x_2]/(f)$. For example, if $f = x_2 - x_1^2$, then $\mathbb{F}[X] = \mathbb{F}[x_1, x_2]/(x_2 - x_1^2) \xrightarrow{\sim} \mathbb{F}[x_1]$, the same as the algebra of functions on $\mathbb{F}$ viewed as an algebraic subset of itself.

3) Geometric significance of localization.

3.1) Localizing one element.

Let $X \subset \mathbb{F}^n$ be an algebraic subset & $f \in \mathbb{F}[X]$. We want to find a geometric interpretation of the localization $\mathbb{F}[X][f^{-1}]$.

Let $f_1, \dots, f_m$ be generators of $I(X) \Rightarrow \mathbb{F}[X] = \mathbb{F}[x_1, \dots x_n]/(f_1, \dots f_m)$. Lemma in Sec 1.1 of Lec 9 tells us that

$$\mathbb{F}[X][f^{-1}] \simeq \mathbb{F}[X][t]/(tf-1) = \mathbb{F}[x_1, \dots x_n, t]/(f_1, \dots, f_m, tf-1).$$

Exercise: Show that if A is an algebra w/o nonzero nilpotent elements, then any localization of A has no nonzero nilpotent elements.

So $\mathbb{F}[X][f^{-1}]$ has no nonzero nilpotent elements $\Leftrightarrow$ the ideal $(f_1,\dots,f_m, tf-1)$ is radical. The corresponding algebraic subset of $\mathbb{F}^{n+1}$ is $\{(\alpha_1,\dots,\alpha_n,z)\in\mathbb{F}^{n+1} \mid f_i(\alpha_1,\dots\alpha_n)=0\ \forall i=1,\dots m;\ zf(\alpha_1,\dots\alpha_n)=1\}$

The projection $\mathbb{F}^{n+1}\twoheadrightarrow\mathbb{F}^n$ forgetting the z coordinate identifies this algebraic subset w $\{\alpha\in X \mid f(\alpha)\neq 0\}$. Denote this subset by X_f. We note that it's <u>not</u> an algebraic subset of $\mathbb{F}^n$ in our conventions. The subset $X_f\subset X$ is called a principal open subset.

Here's an explanation of the terminology.

Definition: • a subset $Y\subset X$ is called Zariski closed if it's an algebraic subset of $\mathbb{F}^n$.

• A subset $U\subset X$ is Zariski open if $X\setminus U$ is Zariski closed.

Example: $X_f\subset X$ is Zariski open b/c $X\setminus X_f=\{\alpha\in X \mid f(\alpha)=0\}$ is closed.

Exercise: Any Zariski open subset of X is the union of, in fact, finitely many, principal open subsets.

Remark: Zariski open/closed subsets are open/closed subsets in a topology (called the Zariski topology). Principal open subsets form a "base of topology."