Lecture 16: Categories, functors & functor morphisms, I

1) Geometric significance of localization, cont'd.
2) Categories.

Ref: [E], Intro to Sec 2; [R], Section 1.1
BONUS: Homotopy category of topological space.

1) Geometric significance of localization, cont'd.

Let $X \subset \mathbb{F}^n$ be algebraic subset, $A := \mathbb{F}[X]$, $f \in A$. In Sec 3 of Lec 15 we have interpreted $A[f^{-1}]$ as the algebra of polynomial functions on $X_f := \{\alpha \in X \mid f(\alpha) \neq 0\}$ embedded as an algebraic subset $\{(\alpha, z) \mid \alpha \in X, zf(\alpha) = 1\} \subset \mathbb{F}^n$ We called X_f a principal open subset.

Now let $\mathfrak{m} \subset A$ be a maximal ideal. Recall that we write $A_{\mathfrak{m}}$ for $A[(A \setminus \mathfrak{m})^{-1}]$.

Note that $A_{\mathfrak{m}}$ is not fintely generated (in general) so is not the algebra of functions of an algebraic subset. It still has a geometric meaning that we are going to discuss now.

For simplicity, assume X is irreducible $\Leftrightarrow I(X)$ is prime $\Leftrightarrow$ $A = \mathbb{F}[x_1, \dots x_n]/I(X)$ is a domain $\rightsquigarrow$ fraction field $\operatorname{Frac}(A)$ $= \{\frac{f}{g} \mid f, g \in A, g \neq 0\}$. By Sec 2.3 of Lec 8, $A_{\mathfrak{m}} = \{\frac{f}{g} \in \operatorname{Frac}(A) \mid g \notin \mathfrak{m}\}$.

By Exercise 3) in Sec 2 of Lec 15, $\exists! \alpha \in X$ s.t. $\mathfrak{m} = \{f \in \mathbb{F}[X] \mid f(\alpha) = 0\}$.

$$\text{So } A_{\mathfrak{m}} = \{\tfrac{f}{g} \mid g(\alpha) \neq 0\} = \bigcup_{g \mid g(\alpha) \neq 0} A[g^{-1}] = \bigcup_{g \mid g(\alpha) \neq 0} \mathbb{F}[X_g]$$

Conclusion:

Every element of $A_{\mathfrak{m}}$ is a function on a principal open subset containing α, but which subset we choose depends on this element.

Remark: When X is reducible, the conclusion still holds but $A_{\mathfrak{m}} = \bigcup_g \mathbb{F}[X_g]$ makes no sense b/c $\mathbb{F}[X_g] = A[g^{-1}]$ are not subrings in a fixed ring (in general). To fix this one replaces the union w. the "direct limit."

Remark (on terminology): Recall (Problem 2 in HW6) that $A_{\mathfrak{m}}$ has the unique maximal ideal $(\mathfrak{m}_{\mathfrak{m}})$ and that rings satisfying this condition are called "local." Conclusion gives a geometric justification for the terminology: the algebra $A_{\mathfrak{m}}$ controls what happens locally (in the Zariski topology) near $\alpha \in X$.

2) Categories.

We proceed to the 2nd part of this course: Category theory. On the one hand it gives language that much of Mathematics uses. On the other hand, it has rich interactions w. Commutative algebra. For our purposes, general constructions of Category theory can be used to motivate more concrete constructions in Commutative algebra (although chronologically it was the other way around).

Our exposition of Category theory will start w. exploring basic notions: categories, functors & functor morphisms.

Definitions below will have a familiar structure: have data & axioms. E.g. here's a basic algebraic structure.

Definition: a monoid is

(Data): a set M equipped w. multiplication map $M \times M \to M$

(axioms): that is associative and has unit, 1.

For example, a group is exactly a monoid, where all elements are invertible. Every ring is a monoid w.r.t. multiplication.

2.1) Definition of a category.

Definition: A category, $\mathcal{C}$, consists of

(Data): • a "collection" of objects, $Ob(\mathcal{C})$.

• $\forall X, Y \in Ob(\mathcal{C}) \rightsquigarrow$ a <u>set</u> of morphisms, $Hom_{\mathcal{C}}(X,Y)$

• $\forall X, Y, Z \in Ob(\mathcal{C})$, a map (of sets) called composition

$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{C}}(X,Z),\ (f,g) \mapsto g \circ f$$

($\circ$ is often omitted from the notation).

These satisfy:

(Axioms): i) composition is associative:

$(f \circ g) \circ h = f \circ (g \circ h)\ \forall\ f \in Hom_{\mathcal{C}}(W,X),\ g \in Hom_{\mathcal{C}}(X,Y),\ h \in Hom_{\mathcal{C}}(Y,Z)$.

ii) Units: $\forall X \in Ob(\mathcal{C})\ \exists\ 1_X \in Hom_{\mathcal{C}}(X,X)$ s.t.

- $f \circ 1_X = f \quad \forall f \in Hom_e(X,Y)$,
- $1_X \circ g = g \quad \forall g \in Hom_e(Z,X)$.

2.2) Examples

1) Category of sets, Sets: objects = sets, morphisms = maps of sets, composition = composition of maps. Axioms: classical (unit $1_X = id_X$).

2) Sets w. additional structure: objects = sets w. this structure morphisms = maps compatible with this structure, composition = = composition of maps.

These include

a) Category of groups, Groups: objects are groups, morphisms = homomorphisms of groups.

b) Category of rings, Rings.

c) For a ring A, have categories of A-modules, A-Mod, & A-algebras (A-Alg), in the latter morphisms = A-linear ring homomorphisms

d) Category of algebraic subsets: the objects are algebraic subsets in $\mathbb{F}^n$ (for various $n \in \mathbb{Z}_{\geq 0}$, here $\mathbb{F}$ is an algebraically closed field) & the morphisms are polynomial maps (Problem 5 in HW3).

While the categories where objects are sets w. additional structure & morphisms are maps w. usual composition occur frequently, it's important to remember that a general category is NOT of this form. Here's a basic example.

3) Note: $\forall X \in Ob(\mathcal{C}) \Rightarrow Hom_{\mathcal{C}}(X,X)$ is a monoid w.r.t. $\circ$. Conversely, every monoid, M, gives a category w. one object, X, & $(Hom_{\mathcal{C}}(X,X), \circ) := M$.

In the Bonus section we'll sketch a very important example from Topology: the <u>homotopy</u> category of topological spaces, where objects are sets w. additional structure but morphisms are equivalence classes of maps.

2.3) Remarks:

1) Sometimes, objects in a category form a set (here we say our category is small). In general, they form a "class", a notion defined in Set theory. We'll ignore this issue (e.g. because, while the categories we are working with are not small, they are "essentially small").

2) $1_X \in Hom_{\mathcal{C}}(X,X)$ is uniquely determined. Moreover, if $f \in Hom_{\mathcal{C}}(X,Y)$ has a (2-sided) inverse g (i.e. $g \in Hom_{\mathcal{C}}(Y,X) \mid f \circ g = 1_Y, g \circ f = 1_X$) then g is unique, $f^{-1} := g$. In this case, f is called an isomorphism; we say X & Y are isomorphic (X & Y behave the same from the point of view of $\mathcal{C}$, e.g. $Z \in Ob(\mathcal{C}) \rightsquigarrow$

$$\mathrm{Hom}_{\mathcal{C}}(Z,X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z,Y)$$
$$\psi \longmapsto f\circ\psi \qquad \text{(inverse is } \psi' \mapsto f^{-1}\circ\psi'\text{)}.$$

Notation: $X \xrightarrow{f} Y$ means $f\in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

2.4) Subcategories: $\mathcal{C}$ is a category.

Def'n: (i) By a subcategory, $\mathcal{C}'$, in $\mathcal{C}$ we mean:

(Data) • A subcollection, $Ob(\mathcal{C}')$, in $Ob(\mathcal{C})$.

• $\forall\, X,Y\in Ob(\mathcal{C}')$, a subset $\mathrm{Hom}_{\mathcal{C}'}(X,Y)\subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ s.t.

(Axioms) • If $f\in \mathrm{Hom}_{\mathcal{C}'}(X,Y)$, $g\in \mathrm{Hom}_{\mathcal{C}'}(Y,Z) \Rightarrow g\circ f\in \mathrm{Hom}_{\mathcal{C}'}(X,Z)$

• $1_X\in \mathrm{Hom}_{\mathcal{C}'}(X,X)$ $\forall\, X\in Ob(\mathcal{C}')$.

(ii) A subcategory $\mathcal{C}'$ in $\mathcal{C}$ is called full if $\mathrm{Hom}_{\mathcal{C}'}(X,Y)=\mathrm{Hom}_{\mathcal{C}}(X,Y)$, $\forall\, X,Y\in Ob(\mathcal{C}')$.

A subcategory $\mathcal{C}'$ has a natural category structure.

Examples:

1) A monoid M = category w. one object

A nonempty subcategory M' in M = a submonoid.

M' is full $\Leftrightarrow$ $M'=M$.

2) $\mathbb{Z}$-Mod (a.k.a. category of abelian groups) is a full subcategory in Groups

3) The category of commutative rings, CommRings is a full subcategory in Rings.

2.5) Constructions w. categories.

Definition 1: For categories $\mathcal{C}_1, \mathcal{C}_2$, their product $\mathcal{C}_1 \times \mathcal{C}_2$ is defined by:
- $Ob(\mathcal{C}_1 \times \mathcal{C}_2) = Ob(\mathcal{C}_1) \times Ob(\mathcal{C}_2)$
- $Hom_{\mathcal{C}_1 \times \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) = Hom_{\mathcal{C}_1}(X_1, Y_1) \times Hom_{\mathcal{C}_2}(X_2, Y_2)$
- composition is componentwise.

Definition 2: For a category, $\mathcal{C}$, its opposite category, $\mathcal{C}^{opp}$ consists of
- the same objects as $\mathcal{C}$,
- $Hom_{\mathcal{C}^{opp}}(X, Y) := Hom_{\mathcal{C}}(Y, X)$
- $g \circ^{opp} f := f \circ g$ $(f \in Hom_{\mathcal{C}^{opp}}(X, Y) = Hom_{\mathcal{C}}(Y, X),$
$g \in Hom_{\mathcal{C}^{opp}}(Y, Z) = Hom_{\mathcal{C}}(Z, Y))$.

Note that $(\mathcal{C}^{opp})^{opp} = \mathcal{C}$.

Example: Let $\mathcal{C}$ be the category of algebraic subsets (2d in Sec 1.2). We are going to describe its opposite based on Problem 5 in HW3: Consider the category $\mathcal{D}$, where
- objects are finitely generated $\mathbb{F}$-algebras w/o nonzero nilpotent elements & together w. a system of generators (cf. Sec 2 in Lec 15)
- and morphisms are $\mathbb{F}$-algebra homomorphisms (w. usual composition of maps).

We can identify $\mathcal{D}$ w. $\mathcal{C}^{opp}$: $Ob(\mathcal{D}) = Ob(\mathcal{C})$ by Rem in Sec 2 of Lec 15 and the identification of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ w. $\mathrm{Hom}_{\mathcal{D}}(Y,X)$ is Problem 5 in HW 3. Details are left as an exercise.

Note that the description of $\mathcal{D}$ doesn't look very natural. First, in the definition of objects we have a collection of generators, which play no further role. Removing this from the defintion of $\mathcal{D}$, we get the category $\mathcal{D}^{opp}$ of "affine varieties/$\mathbb{F}$". The choice of generators of the algebra $\mathbb{F}[X]$ of polynomial functions on an algebraic subset $X\subset\mathbb{F}^n$ corresponded to the inclusion $X\subset\mathbb{F}^n$ so "affine varieties" should be thought of as "algebraic subsets irrespective of an embedding into $\mathbb{F}^n$".

Second, we impose conditions on an algebra that can, in principle, be removed. If we don't require absense of nonzero nilpotent elements then $\mathcal{D}^{opp}$ becomes the category of "finite type affine schemes/$\mathbb{F}$", while removing the condition that the algebra is finitely generated we get the category of "affine schemes over $\mathbb{F}$."

BONUS: homotopy category of topological spaces.

We would like to sketch an important example of a category where objects are sets w. additional structure but morphisms aren't maps - rather they are equivalence classes of maps.

B1) Equivalence on morphisms.

Let $\mathcal{C}$ be a category. Suppose that $\forall X,Y\in Ob(\mathcal{C})$, the set

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is endowed with an equivalence relation $\sim$ s.t.
(1) If $g, g' \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ are equivalent & $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, then $g \circ f \sim g' \circ f$.
(2) If $f, f' \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ are equivalent and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, then $g \circ f \sim g \circ f'$.

We write $[f]$ for the equivalence class of f.
Given such an equivalence relation, we can form a new category to be denoted by $\mathcal{C}/\sim$ as follows:
- $\mathrm{Ob}(\mathcal{C}/\sim) := \mathrm{Ob}(\mathcal{C})$
- $\mathrm{Hom}_{\mathcal{C}/\sim}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/\sim$ - the set of equivalence classes
- $[g] \circ [f] = [g \circ f]$ - well-defined precisely b/c of (1) & (2)

We note that there is a natural functor $\pi : \mathcal{C} \to \mathcal{C}/\sim$ given by $X \mapsto X$, $f \mapsto [f]$.

Example: Let M be a monoid. Note that the equivalence class of $1 \in M$ is a submonoid, say M_0, moreover, (1) & (2) imply that $mM_0 = M_0 m$ $\forall$ $m \in M$. Such submonoids are called normal (for groups we recover the usual condition). And if $M_0 = [1]$ is normal, then (1) and (2) hold - an exercise. For a normal submonoid M_0 we can M/M_0 with a natural monoid structure - just as we do for groups. The category $\mathcal{C}/\sim$ corresponds to the quotient monoid M/M_0 and the functor π is just the natural epimorphism $M \twoheadrightarrow M/M_0$.

Rem*: $\mathcal{C}/\sim$ looks like a quotient category. But in situations where the term "quotient" is used and that are closer to quotients of abelian groups (Serre quotients of abelian categories) the construction is different - and more difficult.

B2) Homotopy category of topological spaces.

Let's recall the usual category of topological spaces. Let X be a set. One can define the notion of topology on X: we declare some subsets of X to be "open", these are supposed to satisfy certain axioms. A set w. topology is called a topological space. A map $f: X \to Y$ of topological spaces is called continuous if $U \subset Y$ is open $\Rightarrow f^{-1}(U) \subset X$ is open.
We define the category Top of topological spaces w.

$Ob(Top)$ = topological spaces.

$Hom_{Top}(X,Y) :=$ continuous maps $X \to Y$

Composition = composition of maps.

One issue: this category is hard to understand - hard to study topological spaces up to homeomorphisms.

Now we introduce our equivalence relation of $Hom_{Top}(X,Y)$

Definition: Continuous maps $f_0, f_1 : X \to Y$ are called homotopic if $\exists$ a continuous map $F: X \times [0,1] \to Y$ s.t. $f_0(x) = F(x,0)$ &

$f_1(x) = F(x,1)$.

Informally, f_0, f_1 are homotopic if one can continuously deform f_0 to f_1. It turns out that being homotopic is an equivalence relation satisfying (1) & (2) from B1. The corresponding category $Top/\sim$ is known as the homotopy category of topol'l spaces. Note that in this category morphisms are <u>not</u> maps!

Here is why we care about the homotopy category. Isomorphic here means homotopic (X is homotopic to Y if $\exists$ $X \xrightarrow{f} Y$, $Y \xrightarrow{g} X$ s.t. fg is homotopic to 1_Y & gf is homotopic to 1_X) and this is easier to understand than being homeomorphic. Second, the classical invariants such as homology and homotopy groups only depend on homotopy type. A more educated way to state this:
these invariants are "functors" from the homotopy category of topological spaces to Groups (true as stated for homology, for homotopy it's more subtle, this requires fixing a point in X and hence need to work w. an auxiliary category of "pointed" topological spaces - up to homotopy).