Lecture 17: Categories, functors & functor morphisms, II.

1) Functors.

2) Functor morphisms.

Ref: [R], Sec. 1.3, 1.4.

1) Functors

Motto: a relation between a category and a functor generalizes a relation between a monoid & a homomorphism.

1.1) Definition

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition: A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is

(Data) • an assignment $X \mapsto \mathcal{F}(X): Ob(\mathcal{C}) \to Ob(\mathcal{D})$.

• $\forall X, Y \in Ob(\mathcal{C})$, a map
$$\begin{array}{ccc} Hom_{\mathcal{C}}(X,Y) & \longrightarrow & Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \mathcal{F}(f) \end{array}$$

(Axioms) – compatibility between compositions & units

• $\forall f \in Hom_{\mathcal{C}}(X,Y), g \in Hom_{\mathcal{C}}(Y,Z) \Rightarrow \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$

equality in $Hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Z))$.

• $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ $\forall X \in Ob(\mathcal{C})$

Example: Let $\mathcal{C}, \mathcal{D}$ be categories w. single object corresponding to monoids M, N (Example 3) in Sec 2.2 of Lec 16). A functor $\mathcal{C} \to \mathcal{D}$ is the same thing as a monoid homomorphism.

Remarks: i) Have the identity functor $Id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$

ii) For functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$, $\mathcal{G}: \mathcal{D} \to \mathcal{E}$ can take the composition $\mathcal{G} \circ \mathcal{F}: \mathcal{C} \to \mathcal{E}$ $(\mathcal{G} \circ \mathcal{F}(X) = \mathcal{G}(\mathcal{F}(X)), \mathcal{G} \circ \mathcal{F}(f) = \mathcal{G}(\mathcal{F}(f)))$ it's a functor (exercise).

iii) A functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is the same thing as a functor $\mathcal{C}^{opp} \to \mathcal{D}^{opp}$.

1.2) Examples of functors

1) Let $\mathcal{C}'$ be a subcategory in $\mathcal{C}$. Then have the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ sending objects/morphisms in $\mathcal{C}'$ to the same objects/morphisms now in $\mathcal{C}$; axioms are clear.

2) Forgetful functors: forget a part of the structure

2a) For: Groups $\to$ Sets;

On objects: For$(G) = G$ viewed as a set.

On morphisms: For$(f) = f$, viewed as a map of sets.

Axioms: clear.

2b) Let A be a commutative ring. Then have the forgetful functor For: A-Alg $\longrightarrow$ A-Mod, forgetting the ring multiplication.

2c) Let A, B be commutative rings & $\varphi: A \to B$ be a ring homom'm. Then we can consider the pullback functor $\varphi^*: B\text{-Mod} \to A\text{-Mod}$. It sends $M \in Ob(B\text{-Mod})$ to M viewed as an A-module & $\psi \in Hom_B(M,N)$

to $\psi \in \mathrm{Hom}_A(M,N)$. Forgets part of the action.

3) Let $\mathcal{C}$ be a category. For $X \in Ob(\mathcal{C})$ define the Hom functor $F_X \ (:= \mathrm{Hom}_{\mathcal{C}}(X,\cdot)): \mathcal{C} \longrightarrow Sets$.

On objects: $F_X(Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)$, a set.

On morphisms: $Y_1 \xrightarrow{f} Y_2 \rightsquigarrow$ map

$$\begin{array}{ccc} F_X(f): \mathrm{Hom}_{\mathcal{C}}(X,Y_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X,Y_2) \\ \psi & \longmapsto & f \circ \psi \end{array}$$

Check axioms: composition: $F_X(g \circ f) = F_X(g) \circ F_X(f)$ for $Y_1 \xrightarrow{f} Y_2 \xrightarrow{g} Y_3$. For $\psi \in \mathrm{Hom}_{\mathcal{C}}(X,Y_1)$ have

$[F_X(g \circ f)](\psi) = (g \circ f) \circ \psi \in \mathrm{Hom}_{\mathcal{C}}(X,Y_3)$.

$[F_X(g) \circ F_X(f)](\psi) = [F_X(g)](f \circ \psi) = g \circ (f \circ \psi)$.

By associativity axiom for morphisms, the two coincide.

The unit axiom is left as exercise.

3^{opp}) We can apply this construction to $\mathcal{C}^{opp} \rightsquigarrow$

$F_X^{opp}: Y \longmapsto \mathrm{Hom}_{\mathcal{C}^{opp}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$

$f \in \mathrm{Hom}_{\mathcal{C}^{opp}}(Y_1,Y_2) = \mathrm{Hom}_{\mathcal{C}}(Y_2,Y_1) \rightsquigarrow$

$$\begin{array}{ccc} F_X^{opp}(f): \mathrm{Hom}_{\mathcal{C}}(Y_1,X) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Y_2,X) \\ \psi & \longmapsto & \psi \circ f \end{array}$$ – map of sets

We can view F_X^{opp} as a functor $\mathcal{C} \longrightarrow Sets^{opp}$ (cf. Remark (iii) in Sec 1.1).

A traditional name: contravariant functor $\mathcal{C} \to Sets$.

4) Algebra constructions as functors:

4a) The "free" functor:

Let A be a ring. Want to define a functor Free: Sets $\to$ A-Mod

I, set, $\rightsquigarrow$ Free $(I) := A^{\oplus I}$

$f: I \to J \rightsquigarrow$ Free $(f): A^{\oplus I} \to A^{\oplus J}$ - the unique A-linear map sending the basis element e_i $(i \in I)$ to $e_{f(i)} \in A^{\oplus J}$.

Checking axioms of functor: exercise.

4b) Localization of modules is a functor: $S \subset A$ multiplicative $\rightsquigarrow \bullet[S^{-1}]: A\text{-Mod} \to A[S^{-1}]\text{-Mod}$, a functor that sends an A-module M to the $A[S^{-1}]$-module $M[S^{-1}]$ and an A-module homomorphism $\varphi: M \to N$ to $\varphi[S^{-1}]: M[S^{-1}] \to N[S^{-1}]$ (see Sec 2.2 of Lec 9) $\varphi[S^{-1}](\frac{m}{s}) := \frac{\varphi(m)}{s}$. Checking the axioms was a part of the very important exercise there

2) Functor morphisms.

Motto: A relation between functors & functor morphisms is like a relation between modules & module homomorphisms.

2.1) Definition: Let $\mathcal{C}, \mathcal{D}$ be categories & $F, G: \mathcal{C} \to \mathcal{D}$ be functors.

Def'n: A **functor morphism** $\eta: F \Rightarrow G$ is

Functors F, G send the objects $X \in Ob(\mathcal{C})$ to $F(X), G(X) \in Ob(\mathcal{D})$.

We can relate $F(X), G(X)$ by taking a morphism between them:

(Data) $\forall X \in Ob(\mathcal{C})$, a morphism $\eta_X \in Hom_{\mathcal{D}}(F(X), G(X))$

Picking morphisms which are totally unrelated is pointless. We need to relate η_X, η_Y for $X, Y \in Ob(\mathcal{C})$. The relations we need come from morphisms between X, Y: $f \in Hom_{\mathcal{C}}(X,Y) \rightsquigarrow$ $F(f) \in Hom_{\mathcal{D}}(F(X), F(Y))$, $G(f) \in Hom_{\mathcal{D}}(G(X), G(Y))$

(axiom) s.t. $\forall\ X, Y \in Ob(\mathcal{C})$, $f \in Hom_{\mathcal{C}}(X,Y)$, the following diagram is commutative:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

Example: Let $\mathcal{C} = Groups$, $\mathcal{D} = Sets$ & $F = G$ is the forgetful functor. Let $n \in \mathbb{Z}$. We are going to construct $\eta_n : F \Rightarrow F$. For a group H, set $\eta_{n,H} : H \to H$ (map of sets), $h \mapsto h^n$. The axiom is satisfied: we need to check that for every group homomorphism $\tau : H \to H'$ the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "\tau"] \arrow[d, "h \mapsto h^n"] & H' \arrow[d, "h' \mapsto h'^n"] \\ H \arrow[r, "\tau"] & H' \end{tikzcd}$$

$\tau(h)^n = \tau(h^n)$

τ is homomorphism.

Remarks:

1) In many (but not all) examples, η_X is "natural" meaning it's "uniform" & "independent of additional choices". Hence the name "natural transformation" for a functor morphism that was used in the past.

2) An analogy w. module homomorphisms is as follows. Let A be a ring, M, N be A-modules. For $a \in A$, we write

a_M, a_N for the operators of multiplication by a in M, N. Then a group homomorphism $\eta: M \to N$ is an A-module homomorphism iff $\forall\, a \in A$, the following is commutative:

$$\begin{tikzcd} M \arrow[r, "a_M"] \arrow[d, "\eta"] & M \arrow[d, "\eta"] \\ N \arrow[r, "a_N"] & N \end{tikzcd}$$

Here elements of A are analogs of elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ & M, N are analogs of functors

Exercise: Let M, N be categories w. one object a.k.a. monoids & $F: M \to N$ be a functor (a.k.a. monoid homomorphism). Then a functor endomorphism $\eta: F \Rightarrow F$ is the same thing as an element $\eta \in N \mid \eta F(m) = F(m)\eta \;\; \forall\, m \in M$.

2.2) Important example.

Let $X, X' \in Ob(\mathcal{C}) \rightsquigarrow$ functors

$$F_X := \mathrm{Hom}_{\mathcal{C}}(X, \bullet),\; F_{X'} := \mathrm{Hom}_{\mathcal{C}}(X', \bullet) : \mathcal{C} \to \mathit{Sets}.$$

Goal: from $g \in \mathrm{Hom}_{\mathcal{C}}(X', X)$ produce a functor morphism $\eta^g: F_X \Rightarrow F_{X'}$ (note that the order is swapped)

i.e. for each $Y \in Ob(\mathcal{C})$ we need to define a map

$$\eta^g_Y: \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X', Y): \qquad X' \xrightarrow{g} X \xrightarrow{\psi} Y$$

$$\psi \longmapsto \psi \circ g$$ $\leftarrow$ essentially the only natural way to give such a map.

Now we need to check the axiom (commutative diagram):
$\forall f \in Hom_{\mathcal{C}}(Y_1, Y_2)$, $F_X(f) = f\circ ?$, $F_{X'}(f) = f \circ ?$ & we have that

$$\left.\begin{tikzcd}
\varphi \in Hom_{\mathcal{C}}(X, Y_1) \arrow[r, "f\circ ?"'] \arrow[d, "\eta^g_{Y_1}(?) = ?\circ g"] & Hom_{\mathcal{C}}(X, Y_2) \arrow[d, "\eta^g_{Y_2}(?) = ?\circ g"] \\
Hom_{\mathcal{C}}(X', Y_1) \arrow[r, "f\circ ?"] & Hom_{\mathcal{C}}(X', Y_2)
\end{tikzcd}\right\}\text{ is commutative}$$

$\downarrow \rightarrow$: $\varphi \mapsto \varphi\circ g \mapsto f\circ(\varphi\circ g)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$ $\longleftarrow$ b/c composition in a category is associative.

$\rightarrow \downarrow$: $\varphi \mapsto f\circ\varphi \mapsto (f\circ\varphi)\circ g$

We've checked: η^g is a functor morphism.

2.3) Remarks

1) Can compose functor morphisms: $\eta: F \Rightarrow G$, $\xi: G \Rightarrow H \rightsquigarrow \xi\circ\eta: F \Rightarrow H$: $(\xi\circ\eta)_X := \xi_X \circ \eta_X$. To check this is a morphism is an exercise. Also have the identity morphism $1_F: F \Rightarrow F$. This allows us to talk about functor isomorphisms: functor morphisms w. two-sided inverse. Not'n for isomorphisms: $F \xRightarrow{\sim} G$. The following is very important:

Exercise: Let $F, G: \mathcal{C} \to \mathcal{D}$ be functors. For $\eta: F \Rightarrow G$ TFAE
1) η is an isomorphism
2) $F(X) \xrightarrow{\eta_X} G(X)$ is an isomorphism in $\mathcal{D}$ $\forall X \in Ob(\mathcal{C})$

Moreover, we have $(\eta^{-1})_X = (\eta_X)^{-1}$.