Lecture 18: categories, functors & functor morphisms, III.

1) Yoneda lemma and applications.

2) Products in categories.

Refs: [R], Secs 2.1-2.3; [HS], Sec II.5.

BONUS: category equivalences.

1) Yoneda lemma and applications

Let $\mathcal{C}$ be a category. Recall that to $X \in Ob(\mathcal{C})$ we can assign a functor $F_X: \mathcal{C} \to Sets$ sending $Y \in Ob(\mathcal{C})$ to $Hom_{\mathcal{C}}(X,Y)$ & a morphism $Y_1 \xrightarrow{f} Y_2$ to the map $Hom_{\mathcal{C}}(X,Y_1) \longrightarrow Hom_{\mathcal{C}}(X,Y_2)$, $\psi \mapsto f\circ\psi$.

Next, for $g \in Hom_{\mathcal{C}}(X',X)$ we have a functor morphism

$\eta^g: F_X \Rightarrow F_{X'}: \eta^g_Y: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{C}}(X',Y)$, $\eta^g_Y(\psi) := \psi\circ g$.

The following foundational result is extremely important

Thm (Yoneda lemma): Every functor morphism $\eta: F_X \Rightarrow F_{X'}$ is of the form η^g for unique $g \in Hom_{\mathcal{C}}(X',X)$.

1.1) Proof of Yoneda lemma:

Step 1: η gives $\eta_X: Hom_{\mathcal{C}}(X,X) = F_X(X) \longrightarrow F_X(X') = Hom_{\mathcal{C}}(X',X)$

Set $g_\eta := \eta_X(1_X)$. We need to show:

- $g_{\eta^g} = g$ &
- $\eta^{g_\eta} = \eta$.

Step 2: $g_{\eta g} = [\eta^g]_X(1_X) = 1_X \circ g = g$.

Step 3: We show that $\eta^{g_\eta} = \eta \iff \forall\, Y \in Ob(\mathcal{C})$ have $(\eta^{g_\eta})_Y = \eta_Y$ an equality of maps $Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{C}}(X',Y)$

Note that $(\eta^{g_\eta})_Y$ sends $f \in Hom_{\mathcal{C}}(X,Y)$ to $f \circ \eta_X(1_X)$.

Now recall that η satisfies the following: $\forall\, Y, Y' \in Ob(\mathcal{C})$ & $Y' \xrightarrow{f} Y$ the diagram below is commutative

$$\begin{tikzcd}
Hom_{\mathcal{C}}(X,Y') \arrow[r, "F_X(f) = f\circ ?"] \arrow[d, "\eta_{Y'}"] & Hom_{\mathcal{C}}(X,Y) \arrow[d, "\eta_Y"] \\
Hom_{\mathcal{C}}(X',Y') \arrow[r, "F_{X'}(f) = f\circ ?"] & Hom_{\mathcal{C}}(X',Y)
\end{tikzcd}$$

We use this for $Y' = X$, and apply the equal morphisms to 1_X.

$$\begin{tikzcd}
1_X \in Hom_{\mathcal{C}}(X,X) \arrow[r, "f\circ ?"] \arrow[d, "\eta_X"] & Hom_{\mathcal{C}}(X,Y) \arrow[d, "\eta_Y"] \\
Hom_{\mathcal{C}}(X',X) \arrow[r, "f\circ ?"] & Hom_{\mathcal{C}}(X',Y)
\end{tikzcd}$$

$\downarrow \longrightarrow$: $f \circ \eta_X(1_X) = (\eta^{g_\eta})_Y(f) \in Hom_{\mathcal{C}}(X',Y)$

$\longrightarrow \downarrow$: $\eta_Y(f \circ 1_X) = \eta_Y(f) \in Hom_{\mathcal{C}}(X',Y)$

The equality $(\eta^{g_\eta})_Y = \eta_Y$ follows finishing the proof. $\square$

1.2) Yoneda lemma vs compositions & isomorphisms

Lemma: 1) Let $X, X', X'' \in Ob(\mathcal{C})$, $X \xrightarrow{g} X' \xrightarrow{g'} X''$ be morphisms yielding $\eta^{g'}: F_{X''} \Rightarrow F_{X'}$, $\eta^{g}: F_{X'} \Rightarrow F_X$. Then $\eta^{g'g} = \eta^{g} \circ \eta^{g'}$

2) η^g is a functor isomorphism $\Leftrightarrow$ the morphism g is isomorphism.

Proof: 1) For $Y \in Ob(\mathcal{C})$, $\psi \in Hom_{\mathcal{C}}(X, Y) \Rightarrow$
$\eta^g_Y \circ \eta^{g'}_Y(\psi) = \eta^g_Y(\psi \circ g') = (\psi \circ g') \circ g = \psi \circ (g' \circ g) = \eta^{g'g}_Y(\psi)$.

2) $\Leftarrow$: by 1) $\eta^{g} \circ \eta^{g^{-1}} = \eta^{1_{X'}} = 1_{F_{X'}}$ & similarly in the other direction.
$\Rightarrow$: by Yoneda lemma $\exists!\, h \in Hom_{\mathcal{C}}(X', X)$ w. $\eta^{-1} = \eta^h$. Then $\eta^{1_{X'}}$ $= 1_{F_{X'}} = \eta \circ \eta^{-1} = \eta^{g} \circ \eta^{h} = [1)] = \eta^{gh}$. By the uniqueness part of Thm $gh = 1_{X'}$. Similarly, $hg = 1_X$. $\square$

1.3) Objects representing functors

An important use of the Yoneda lemma is to relate (abstract) constructions in categories & (concrete) constructions w. sets. An important role in this relation is played by the following def'n:

Definition: Let $F: \mathcal{C} \to Sets$ be a functor. We say $X \in Ob(\mathcal{C})$ **represents** F if F is isomorphic to F_X.

A representing object may fail to exist (see an exercise below). If it exists, we say that F is **representable**.

Lemma: An object, X, representing F is unique up to isom'm if it exists.

Proof:

Let $X, X' \in Ob(\mathcal{C})$ represent F: $\eta: F_X \overset{\sim}{\Rightarrow} F \overset{\sim}{\Leftarrow} F_{X'} : \eta' \rightsquigarrow$ functor isomorphism $\eta^{-1} \circ \eta' : F_{X'} \overset{\sim}{\Rightarrow} F_X$. By Yoneda lemma, $\eta^{-1} \circ \eta' = \eta^g$. By 2) of Lemma in Sec 1.2, $X \xrightarrow{g} X'$ is an isomorphism. $\square$

Example: The forgetful functor $For: \mathcal{C} = Groups \to Sets$ is represented by $\mathbb{Z}$. Indeed, for any group G we have a bijection (of sets) $\xi_G : Hom_{Groups}(\mathbb{Z}, G) \xrightarrow{\sim} G$, $\varphi \mapsto \varphi(1)$. (ξ_G) is a functor morphism - what we need to check is that $\forall$ group homomorphism $f: G \to H$, the diagram

$$\begin{tikzcd} Hom_{Groups}(\mathbb{Z}, G) \arrow[r, "\varphi \mapsto f \circ \varphi"] \arrow[d, "\varphi \mapsto \varphi(1)"] & Hom_{Groups}(\mathbb{Z}, H) \arrow[d, "\varphi' \mapsto \varphi'(1)"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

is commutative, which is left as an exercise.

Since ξ_G is bijective $\forall G$, by Sec 2.3 of Lec 17, it's a functor isomorphism, implying our claim.

Exercise: Consider the functor $Rings \to Sets$:

- sending every ring R to a fixed set w. one element.
- and every homomorphism of rings to the identity map.

Show that it's represented by $\mathbb{Z}$.

Exercise: Show that the forgetful functor to sets from the full subcategory in Groups consisting of finite and (for simplicity) abelian groups is not representable.

Example cont'd. We use the isomorphism $F_{\mathbb{Z}} \xrightarrow{\sim} For$ to compute the monoid of all endomorphisms of For. By Yoneda lemma, every endomorphism of $F_{\mathbb{Z}}$ comes from a unique endomorphism of $\mathbb{Z}$ & by Sec 1.2, this bijection is compatible w. compositions (reversing the order). So our job is to compute $Hom_{Groups}(\mathbb{Z},\mathbb{Z})$. As a set, it's $F_{\mathbb{Z}}(\mathbb{Z}) \xrightarrow[\xi_{\mathbb{Z}}]{\sim} For(\mathbb{Z}) = \mathbb{Z}$. The isomorphism $\xi_{\mathbb{Z}}$ sends $\varphi: \mathbb{Z}\to\mathbb{Z}$ to $\varphi(1)\in\mathbb{Z}$ so $m\in\mathbb{Z}$ corresponds to the homomorphism $x\mapsto mx: \mathbb{Z}\to\mathbb{Z}$. composition in $Hom_{Groups}(\mathbb{Z},\mathbb{Z})$ corresponds to the product in $\mathbb{Z}$. It follows that the monoid of functor endomorphisms of $F_{\mathbb{Z}}$ & hence of the isomorphic functor For is $\mathbb{Z}$ w.r.t. multiplication.

What does the endomorphism of For corresponding to $m\in\mathbb{Z}$ (to be denoted by η_m) do? The identification $Hom_{Groups}(\mathbb{Z}, G) = F_{\mathbb{Z}}(G)$ w. G is by $\varphi\mapsto\varphi(1)$. And $\eta_{m,G}: F_{\mathbb{Z}}(G)\longrightarrow F_{\mathbb{Z}}(G)$ sends φ to $[x\mapsto \varphi(mx)=\varphi(x)^m]$. So $\eta_m: G\to G$ is $g\mapsto g^m$, cf. Example in Sec 2.1 of Lec 17.

2) Products in categories.

The concept of a representing object allows to carry constructions from the category of sets to a general category. Here we consider a basic such construction - products.

Recall that in our usual categories: Sets, Groups, Rings, A-mod we have the notion of direct product. In all of them, this is characterized by universal property. E.g. if A_1, A_2 are rings, then $A_1 \times A_2$ is a ring w. ring homomorphisms $\pi_i : A_1 \times A_2 \to A_i$ s.t. $\forall$ rings B w. homomorphisms $\varphi_i : B \to A_i$ $\exists!\ \varphi : B \to A_1 \times A_2$ w. $\varphi_i = \pi_i \circ \varphi$.

Now let $\mathcal{C}$ be a category and $F_1, F_2 : \mathcal{C} \to$ Sets be functors. Define their product $F_1 \times F_2$ by

- Sending $X \in Ob(\mathcal{C})$ to $F_1(X) \times F_2(X)$
- Sending $\varphi \in Hom_{\mathcal{C}}(X,Y)$ to $F_1(\varphi) \times F_2(\varphi) : F_1(X) \times F_2(X) \to F_1(Y) \times F_2(Y)$.

$F_1 \times F_2$ is a functor, to check the axioms is an exercise.

Now take $X_1, X_2 \in Ob(\mathcal{C})$ and let $F_i := F^{opp}_{X_i}$ be the Hom functor $Hom_{\mathcal{C}^{opp}}(X, \bullet)$ $(= Hom_{\mathcal{C}}(\bullet, X)) : \mathcal{C}^{opp} \to$ Sets.

Definition: If $X \in Ob(\mathcal{C})$ represents $F^{opp}_{X_1} \times F^{opp}_{X_2}$, then we say that X is the product $X_1 \times X_2$.

Thx to Lemma in Sec 2, $X_1 \times X_2$ is unique (up to iso) if it exists (that may fail to be the case).

Here's an alternative characterization of products.

Lemma: 1) There are $\pi_i \in Hom_{\mathcal{C}}(X, X_i)$ s.t.

(*) $\forall\ Y \in Ob(\mathcal{C}), \varphi_i \in Hom_{\mathcal{C}}(Y, X_i), i=1,2, \exists!\ \varphi \in Hom_{\mathcal{C}}(Y,X) \mid \varphi_i = \pi_i \circ \varphi$.

2) Conversely, let (X, π_1, π_2) satisfy $(*)$. Then $X = X_1 \times X_2$

Note that $(*)$ is the usual universal property of direct products. In particular, in our usual categories: Sets, Groups, Rings, A-Mod products are just direct products - and they exist $\forall X_1, X_2$.

Proof (of Lemma): 1) Let $\eta: F_X^{opp} \xrightarrow{\sim} F_{X_1}^{opp} \times F_{X_2}^{opp} \rightsquigarrow$ for $Y \in Ob(\mathcal{C})$ $\eta_Y: Hom_{\mathcal{C}}(Y, X) \xrightarrow{\sim} Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2)$. We define $(\pi_1, \pi_2) \in Hom_{\mathcal{C}}(X, X_1) \times Hom_{\mathcal{C}}(X, X_2)$ as $\eta_X(1_X)$. As in Step 3 of the proof of Yoneda lemma in Sec 1.1, $\forall \varphi \in Hom_{\mathcal{C}}(Y, X)$, we have comm've diagram

$$\begin{tikzcd}
Hom_{\mathcal{C}}(X,X) \arrow[r, "?\circ\varphi"] \arrow[d, "\eta_X"', "\wr"] & Hom_{\mathcal{C}}(Y,X) \arrow[d, "\wr"', "\eta_Y"] \\
Hom_{\mathcal{C}}(X,X_1) \times Hom_{\mathcal{C}}(X,X_2) \arrow[r, "(?\circ\varphi,\, ?\circ\varphi)"] & Hom_{\mathcal{C}}(Y,X_1) \times Hom_{\mathcal{C}}(Y,X_2)
\end{tikzcd}$$

which we apply to 1_X getting: $\eta_Y(\varphi) = (\pi_1 \circ \varphi, \pi_2 \circ \varphi)$. $(*)$ follows b/c η_Y is a bijection: $\forall \varphi_1, \varphi_2 \; \exists! \; \varphi$ w. $\eta_Y(\varphi) = (\varphi_1, \varphi_2)$

2) We essentially reverse the argument. Define
$\eta_Y: Hom_{\mathcal{C}}(Y, X) \longrightarrow Hom_{\mathcal{C}}(Y, X_1) \times Hom_{\mathcal{C}}(Y, X_2)$, $\varphi \mapsto (\pi_1 \circ \varphi, \pi_2 \circ \varphi)$
By $(*)$, η_Y is a bijection. To check that η_Y constitute a functor morphism is an exercise. So $\eta = (\eta_Y)$ is a functor isomorphism. $\square$

BONUS: Category equivalences.

Our question here: when are two categories the "same"? Turns out, functor isomorphisms play an important role in answering this question.

Before we address this, we should discuss an easier question: when are two sets the same? Well, they are literally the same if they consist of the same elements. But this definition is quite useless: sets arising from different constructions won't be the same in this sense. Of course, we use isomorphic instead of being literally the same.

Now back to categories. Again, being the same is useless. How about being isomorphic? Turns out, this is not useful. Let's see why. Let $\mathcal{C}, \mathcal{D}$ be categories. We say that $\mathcal{C}, \mathcal{D}$ are isomorphic if there are functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ such that $FG = Id_{\mathcal{D}}$, $GF = Id_{\mathcal{C}}$. The issue is: two functors obtained by different constructions are never the same (compare to sets). The solution: replace "equal" w. "isomorphic" (as functors).

Definition: • Functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ are ==quasi-inverse== if $FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$, $GF \overset{\sim}{\Rightarrow} Id_{\mathcal{C}}$ (isomorphic).

• We say $\mathcal{C}, \mathcal{D}$ are ==equivalent== if there are quasi-inverse functors (called ==equivalences==) $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$.

Now we are going to state a general result. For this we need another definition.

Definitions: A functor $F: \mathcal{C} \to \mathcal{D}$ is called

- fully faithful if $\forall X, X' \in Ob(\mathcal{C}) \Rightarrow$
  $f \mapsto F(f)$ is a bijection $Hom_{\mathcal{C}}(X,X') \xrightarrow{\sim} Hom_{\mathcal{D}}(F(X),F(X'))$
- essentially surjective if $\forall Y \in Ob(\mathcal{D})\ \exists\ X \in Ob(\mathcal{C})$

such that $F(X)$ is isomorphic to Y.

Thm: A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence $\Leftrightarrow$ F is fully faithful & essentially surjective.

We won't prove this, but we will give an example - that illustrates how the proof works in general.

Example: Consider the category $\mathcal{D} = \mathbb{F}\text{-}Vect_{fd}$ of finite dimen'l vector spaces over a field $\mathbb{F}$ and its full subcategory $\mathcal{C}$ w. objects $\mathbb{F}^n$ $(n \geq 0)$. We claim that the inclusion functor $F: \mathcal{C} \hookrightarrow \mathcal{D}$ is an equivalence. It's fully faithful by def'n and it's essentially surjective by the existence of basis.

Now we produce a quasi-inverse functor, G. In each $V \in Ob(\mathcal{D})$ we fix a basis, which leads to an isomorphism $\eta_V : V \xrightarrow{\sim} \mathbb{F}^n$. We define $G(V)$ as $\mathbb{F}^n$. For a linear map $f: U \to V$ (w.

$\dim U = m$, $\dim V = n$) we set $G(f) := \eta_V^{-1} \circ f \circ \eta_U$.

Exercise: Check G is a functor.

Now we are going to simplify our life a bit & assume that $\eta_{\mathbb{F}^n} : \mathbb{F}^n \xrightarrow{\sim} \mathbb{F}^n$ is the identity.

Exercise: $GF : \mathcal{C} \longrightarrow \mathcal{C}$ is the identity functor (not just isomorphic to it).

Now we produce a functor isomorphism $\eta : Id_{\mathcal{D}} \xRightarrow{\sim} FG$
So we need to have $\eta_V : V \to \mathbb{F}^{\dim V}$ and this is the isomorphism from above.

Exercise: prove that η is indeed a functor morphism

Then η is an isomorphism of functors. So F is indeed a category equivalence.

Another exercise: prove that the duality functor $\cdot^*$ is an equivalence $\mathbb{F}\text{ Vect}_{fd} \longrightarrow \mathbb{F}\text{-Vect}_{fd}^{opp}$.