Lecture 19: Categories, functors & functor morphisms IV.

1) Coproducts.
2) Adjoint functors.
BONUS: Adjunction unit & counit.

Refs: [R], Section 4.1; [HS], Sections II.5, II.7

1) Coproducts.

Let $\mathcal{C}$ be a category.

Definition: Let $X_1, X_2 \in Ob(\mathcal{C})$. Their coproduct (that we denote by $X_1 * X_2$) is the product is $\mathcal{C}^{opp}$. I.e.

(I) $F_{X_1 * X_2} \xRightarrow{\sim} F_{X_1} \times F_{X_2}$, where we write F_X for the Hom functor $Hom_{\mathcal{C}}(X, \cdot): \mathcal{C} \to Sets$.

(II) equivalently, there are morphisms $X_i \xrightarrow{\iota_i} X_1 * X_2$, $i=1,2$, s.t. $\forall\ Y \in Ob(\mathcal{C})$ & $X_i \xrightarrow{\varphi_i} Y$, $i=1,2$, $\exists!\ \varphi: X_1 * X_2 \to Y \mid \varphi_i = \varphi \circ \iota_i$

The equivalence of (I) & (II) follows from Lemma in Sec 2 of Lec 18 (where we replace $\mathcal{C}$ w. $\mathcal{C}^{opp}$).

Examples: 1) Let $\mathcal{C} = Sets$. Then $X_1 * X_2 = X_1 \sqcup X_2$ (and ι_i is the natural inclusion). (II) is manifest.

2) Let $\mathcal{C} = A\text{-Mod}$. Then $X_1 * X_2 = X_1 \oplus X_2$: for any A-module Y, have a natural isomorphism

$\eta_Y: \mathrm{Hom}_A(X_1 \oplus X_2, Y) \xrightarrow{\sim} \mathrm{Hom}_A(X_1, Y) \times \mathrm{Hom}_A(X_2, Y)$

see Sec 1 of Lec 4. To check (η_Y) is a functor morphism is an exercise.

Later on we will describe the coproduct in the category of <u>commutative</u> A-algebras (this will be the tensor product).

2) Adjoint functors.

Let $\mathcal{C}, \mathcal{D}$ be categories. Being "adjoint" is the most important relationship that a functor $\mathcal{C} \to \mathcal{D}$ can have with a functor $\mathcal{D} \to \mathcal{C}$.

2.1) Definition

Let $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ be functors.

Definition: F is left adjoint to G (and G is right adjoint to F) if:

$\forall X \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D})$ $\exists$ bijection $\eta_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ s.t.

(1) $\forall X, X' \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D}), X' \xrightarrow{\varphi} X$ $(\rightsquigarrow F(X') \xrightarrow{F(\varphi)} F(X))$ the following is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\eta_{X,Y}", "\sim"'] \arrow[d, "?\circ F(\varphi)"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \arrow[d, "?\circ\varphi"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X'),Y) \arrow[r, "\eta_{X',Y}", "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(X',G(Y))
\end{tikzcd}$$

(2) $\forall\, Y, Y' \in Ob(\mathcal{D}), Y \xrightarrow{\psi} Y', X \in Ob(\mathcal{C})$, the following is comm've

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\eta_{X,Y}", "\sim"'] \arrow[d, "\psi\circ ?"] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \arrow[d, "G(\psi)\circ ?"] \\
\mathrm{Hom}_{\mathcal{D}}(F(X),Y') \arrow[r, "\eta_{X,Y'}", "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y'))
\end{tikzcd}$$

For us the main reason to consider adjoint functors is that we can get interesting functors as adjoints to boring (e.g. forgetful) functors.

2.2) Examples.

Below A is a commutative ring.

Example 1: Let G be $For: A\text{-}Mod \longrightarrow Sets$ (forgetful) $F := Free: Sets \to A\text{-}Mod$ (Example 4a in Sec 1.2 of Lec 17), $Free(I) = A^{\oplus I}$ & for $f: I \to J$ (map of sets): $Free(f)(e_i) = e_{f(i)}$.

Claim: F is left adjoint to G

Below we write $Maps$ for Hom_{Sets} (& Hom_A for $\mathrm{Hom}_{A\text{-}Mod}$)

• construct $\eta_{I,M} : \mathrm{Hom}_A(A^{\oplus I}, M) \xrightarrow{\sim} \mathrm{Maps}(I, M)$

$$\begin{array}{ccc} \mathrm{Hom}_A(A^{\oplus I}, M) & \xrightarrow{\sim} & \mathrm{Maps}(I,M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \tau & \xmapsto[\eta_{I,M}]{} & [i \mapsto \tau(e_i)] \end{array}$$

basis element

• check commutative diagram (1): $\forall$ maps $\varphi : I \to J$

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus J}, M) \arrow[r, "\eta_{J,M}", "\sim"'] \arrow[d, "? \circ \mathrm{Free}(\varphi)"] & \mathrm{Maps}(J, M) \arrow[d, "? \circ \varphi"] \\
\mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\eta_{I,M}", "\sim"'] & \mathrm{Maps}(I, M)
\end{tikzcd}$$

$\downarrow \longrightarrow$: $\tau \mapsto [\text{unique } \tau' : A^{\oplus I} \to M \text{ s.t. } \tau'(e_i) := \tau(e_{\varphi(i)})]$

$$\downarrow$$

$$[i \mapsto \tau(e_{\varphi(i)})]$$

$\longrightarrow \downarrow$: $\tau \mapsto [j \mapsto \tau(e_j)] \mapsto [i \mapsto \tau(e_{\varphi(i)})]$

Check (2): $\forall \ \psi \in \mathrm{Hom}_A(M, N)$, the following is commutative

$$\begin{tikzcd}
\tau \in \mathrm{Hom}_A(A^{\oplus I}, M) \arrow[r, "\eta_{I,M}", "\sim"'] \arrow[d, "\psi \circ ?"] & \mathrm{Maps}(I, M) \arrow[d, "\psi \circ ?"] \\
\mathrm{Hom}_A(A^{\oplus I}, N) \arrow[r, "\eta_{I,N}", "\sim"'] & \mathrm{Maps}(I, N)
\end{tikzcd}$$

where now ψ is viewed as map of sets

Both $\overset{\longrightarrow}{\downarrow}$ & $\downarrow \longrightarrow$ send τ to $[i \mapsto \psi(\tau(i))]$.

The adjunction is established.

Example 2: Let $S \subset A$ be a multiplicative subset $\rightsquigarrow$ localization $A[S^{-1}]$ w. ring homomorphism $\iota: A \longrightarrow A[S^{-1}]$. So we get functors $F := \bullet[S^{-1}]: A\text{-Mod} \to A[S^{-1}]\text{-Mod}$ and $G := \iota^*: A[S^{-1}]\text{-Mod} \longrightarrow A\text{-Mod}$ (pullback = forgetful functor).

Claim: F is left adjoint to G.

For $M \in Ob(A\text{-Mod})$, $N \in Ob(A[S^{-1}]\text{-Mod})$, we have a bijection

$$\eta_{M,N}: \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}], N) \xrightarrow{\sim} \mathrm{Hom}_A(M,N)$$
$$\psi \longmapsto \psi \circ \iota_M$$

(we omit ι^* from the notation)

(where $\iota_M: M \to M[S^{-1}]$, $m \mapsto \frac{m}{1}$), this is 4) of Proposition in Sec 2.2 of Lec 9

Now we need to show that diagrams (1) and (2) from Sec 2.1 commute. Let's check (1): for $\tau \in \mathrm{Hom}_A(M_1, M_2)$ need to show

$$\begin{tikzcd}
\mathrm{Hom}_{A[S^{-1}]}(M_2[S^{-1}], N) \arrow[r, "? \circ \iota_{M_2}"] \arrow[d, "? \circ \tau[S^{-1}]"] & \mathrm{Hom}_A(M_2, N) \arrow[d, "? \circ \tau"] \\
\mathrm{Hom}_{A[S^{-1}]}(M_1[S^{-1}], N) \arrow[r, "? \circ \iota_{M_1}"] & \mathrm{Hom}_A(M_1, N)
\end{tikzcd}$$

commutes

$\to \downarrow$ gives $? \circ \overbrace{\iota_{M_2} \circ \tau}^{M_1 \to M_2[S^{-1}]}$, and $\downarrow \to$ gives $? \circ \overbrace{\tau[S^{-1}] \circ \iota_{M_1}}^{M_1 \to M_2[S^{-1}]}$; for $m \in M_1$, have $\iota_{M_2} \circ \tau(m) = \frac{\tau(m)}{1}$, $\tau[S^{-1}] \circ \iota_{M_1}(m) = \tau[S^{-1}]\left(\frac{m}{1}\right) = \frac{\tau(m)}{1}$. So $\iota_{M_2} \circ \tau = \tau[S^{-1}] \circ \iota_{M_1}$, and the diagram indeed commutes.

Diagram (2) becomes: for $\xi \in \mathrm{Hom}_{A[S^{-1}]}(N_1, N_2)$:

$$\begin{tikzcd}
\mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}], N_1) \arrow[r, "\eta_{M,N_1} = ?\circ\iota_M"] \arrow[d, "\xi\circ ?"] & \mathrm{Hom}_A(M, N_1) \arrow[d, "\xi\circ ?"] \\
\mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}], N_1) \arrow[r, "\eta_{M,N_2} = ?\circ\iota_M"] & \mathrm{Hom}_A(M, N_2)
\end{tikzcd}$$

[$\iota^*(\xi)=\xi$ b/c ι^* is forgetful]

It is commutative.

2.3) Uniqueness.

Proposition: If $F^1, F^2: \mathcal{C}\to\mathcal{D}$ are left adjoint to $G:\mathcal{D}\to\mathcal{C}$, then $F^2 \overset{\sim}{\Rightarrow} F^1$.

Proof: Suppose we have $\eta^i_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F^i(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ that make (1) & (2) comm've $\rightsquigarrow$

$\eta_{XY} := (\eta^2_{X,Y})^{-1}\circ \eta^1_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F^1(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y)$ that make the following analogs of (1) and (2) commutative (exercise)

(1) $\forall\ X' \xrightarrow{\varphi} X$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F^1(X), Y) \arrow[r, "\cancel{\eta_{X,Y}}\ ?\circ\tau_X"] \arrow[d, "?\circ F^1(\varphi)"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y) \arrow[d, "?\circ F^2(\varphi)"] \\
\mathrm{Hom}_{\mathcal{D}}(F^1(X'), Y) \arrow[r, "\cancel{\eta_{X',Y}}\ ?\circ\tau_{X'}"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X'), Y)
\end{tikzcd}$$

(2) $\forall\ Y \xrightarrow{\psi} Y'$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F^1(X), Y) \arrow[r, "\cancel{\eta_{X,Y}}\ ?\circ\tau_X"] \arrow[d, "\psi\circ ?"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y) \arrow[d, "\psi\circ ?"] \\
\mathrm{Hom}_{\mathcal{D}}(F^1(X), Y') \arrow[r, "\cancel{\eta_{X,Y'}}\ ?\circ\tau_X"] & \mathrm{Hom}_{\mathcal{D}}(F^2(X), Y')
\end{tikzcd}$$

Fix X, look at (2): it tells us that $\eta_{X,\bullet}$ is a functor morphism (and hence isomorphism - b/c each $\eta_{X,Y}$ is bijection) between $\mathrm{Hom}_{\mathcal{D}}(F^1(X),\bullet)$ & $\mathrm{Hom}_{\mathcal{D}}(F^2(X),\bullet)$. By Yoneda Lemma, have the unique isomorphism $\tau_X \in \mathrm{Hom}_{\mathcal{D}}(F^2(X), F^1(X))$ s.t. $\eta_{X,Y} = ? \circ \tau_X$. Plug this into diagrams (1) & (2).

We now show that τ is a functor morphism $F^2 \Rightarrow F^1$ (hence an isom'm b/c each τ_X is an iso): we need to show the diagram

$$(*)\qquad \begin{tikzcd} F^2(X') \arrow[r, "\tau_{X'}"] \arrow[d, "F^2(\varphi)"] & F^1(X') \arrow[d, "F^1(\varphi)"] \\ F^2(X) \arrow[r, "\tau_X"] & F^1(X) \end{tikzcd}$$

is commutative. Indeed, (1) is commutative, so

$\psi\circ(\tau_X\circ F^2(\varphi)) = \psi\circ(F^1(\varphi)\circ\tau_{X'})\ \forall\ Y\in Ob(\mathcal{D}),\ \psi\in \mathrm{Hom}_{\mathcal{D}}(F^1(X),Y)$.

Take $Y = F^1(X)$, $\psi = 1_{F^1(X)}$ & get that $(*)$ is commutative. $\square$

2.4) Remarks.

1) Fix X & consider composition of functors

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,\bullet)} Sets$$

If F is left adj't to G, then $F(X)$ represents this composition via isomorphism $\eta_{X,\bullet}$, see Diagram (2) in Sec 2.1.

2) We can view $\mathrm{Hom}_{\mathcal{C}}(\bullet,?)$ as a functor $\mathcal{C}^{opp}\times\mathcal{C} \longrightarrow Sets$

Similarly for $\mathcal{D} \rightsquigarrow$ compositions $\mathcal{C}^{opp} \times \mathcal{D} \longrightarrow Sets$

$\quad Hom_{\mathcal{D}}(F(\cdot), ?)$, $Hom_{\mathcal{C}}(\cdot, G(?))$

Diagrams (1) & (2) combine to show that [F is left adj't to G] $\Longleftrightarrow$ the two functors above are isomorphic (via $\eta_{\cdot,?}$)

3) Many categorical notions (including adjunction) have parallels in Linear Algebra. Let $\mathbb{F}$ be a field. There's a distinguished vector space, $\mathbb{F}$. For a finite dimensional vector space V, we can consider its dual, V^*. Have a vector space pairing $\langle \cdot, \cdot \rangle : V^* \times V \longrightarrow \mathbb{F}$, $\langle \alpha, v \rangle := \alpha(v)$. And for a linear map $A: V \to W$ we can consider its adjoint, the unique linear map $A^*: W^* \to V^*$ s.t. $\langle \beta, Av \rangle = \langle A^*\beta, v \rangle$.

Here are analogs of this for categories. An analog of $\mathbb{F}$ is Sets. An analog of passing from V to V^* is passing from a category $\mathcal{C}$ to the category $\mathcal{C}^{opp}$. An analog of linear maps $U \to V$ is functors $\mathcal{C} \longrightarrow \mathcal{D}$. An analog of the pairing $V^* \times V \longrightarrow \mathbb{F}$ is $Hom_{\mathcal{C}}(\cdot, ?) : \mathcal{C}^{opp} \times \mathcal{C} \longrightarrow Sets$. Finally an analog of $\langle A^*\beta, v \rangle = \langle \beta, Av \rangle$ is our definition of adjoint functors.

There are differences as well. First, a functor $\mathcal{C} \to \mathcal{D}$ is the same thing as a functor $\mathcal{C}^{opp} \to \mathcal{D}^{opp}$ but there's no way to get a linear map $V^* \to W^*$ from $V \to W$. Also adjunction of functors is very sensitive to the sides (the left adjoint of G may not be isomorphic to the right adjoint - moreover exactly one of those may fail to exist), while for linear maps this issue doesn't arise.

BONUS: adjunction unit & counit.

Let $F: \mathcal{C} \to \mathcal{D}$ be left adjoint to $G: \mathcal{D} \to \mathcal{C}$. We claim that this gives rise to functor morphisms: the adjunction unit $\varepsilon: \mathrm{Id}_{\mathcal{C}} \Longrightarrow GF$ & counit $\eta: FG \Longrightarrow \mathrm{Id}_{\mathcal{D}}$.

We construct ε and leave η as an exercise.

Consider $X_1, X_2 \in Ob(\mathcal{C})$. Then we have the bijection

$$\eta_{X_1, F(X_2)}: \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X_1, GF(X_2))$$

Note that F gives rise to a map $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2))$. Composing this map w. the bijection $\eta_{X_1, F(X_2)}$ we get

$$\varepsilon_{X_1, X_2}: \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_1, GF(X_2)).$$

Now we can argue as in the proof of Proposition 1.3 to see that $\exists!\ \varepsilon: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ s.t. $\varepsilon_{X_1, X_2}(\varphi) = \varepsilon_{X_2} \circ \varphi$.

A natural question to ask is: for two functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ & functor morphisms $\varepsilon: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$, $\eta: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ when is F left adjoint to G (& ε, η unit & counit).

Very Premium Exercise: TFAE

a) F is left adjoint to G w. unit ε & counit η

b) The composed morphisms $F \Rightarrow FGF \Rightarrow F$, $G \Rightarrow GFG \Rightarrow G$ induced by ε, η (cf. Problem 8 in HW3) are the identity endomorphisms (of F & G).