Lecture 20: tensor products, I

1) Definition of tensor products of modules.
2) Construction.

Ref: [AM], Section 2.7.

BONUS: Tensor products over noncommutative rings.

0) Roadmap

In the last four lectures we've covered abstract categorical constructions. In the remaining seven lectures we will focus on their interactions w. Commutative algebra. This will split into two topics that are related to each other

1) Tensor products of modules & algebras
2) Exactness, projective and flat modules.

1) Definition of tensor products of modules

1.0) Bilinear maps: Let A be a comm've ring, M_1, M_2, N be A-modules. Recall (Sec 2.3 of Lec 3) that $\beta: M_1 \times M_2 \to N$ is A-bilinear if $\beta(m_1, ?): M_2 \to N$ is A-linear $\forall m_1 \in M_1$ & $\beta(?, m_2): M_1 \to N$ is A-linear $\forall m_2 \in M_2$. Consider the set.

$$\mathrm{Bilin}_A(M_1 \times M_2, N) := \{A\text{-bilinear maps } M_1 \times M_2 \to N\}.$$

Digression: why should we care about bilinear maps –

b/c they are everywhere!

1) Linear algebra: for an $\mathbb{F}$-vector space V can talk about bilinear forms $V\times V\to\mathbb{F}$ (cf. Sec 1.3.1 in Lec 12)

2) if M is an A-module $\Rightarrow$ action map $A\times M\to M$ is A-bilinear.

3) the composition map $\mathrm{Hom}_A(L,M)\times\mathrm{Hom}_A(M,N)\longrightarrow\mathrm{Hom}_A(L,N)$, where L,M,N are A-modules is A-bilinear, Problem 4 in HW1.

4) If B is an A-algebra, then the product map $B\times B\to B$ is A-bilinear.

Observation: $F_{M_1,M_2}:=\mathrm{Bilin}_A(M_1\times M_2,\bullet)$ is actually a functor $A\text{-Mod}\longrightarrow\text{Sets}$: to $\psi\in\mathrm{Hom}_A(N,N')$ we assign

$$\begin{array}{cccc} F_{M_1,M_2}(\psi): & \mathrm{Bilin}_A(M_1\times M_2,N) & \longrightarrow & \mathrm{Bilin}_A(M_1\times M_2,N') \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \beta & \longmapsto & \psi\circ\beta \end{array}$$

Exercise: Show $\psi\circ\beta$ is A-bilinear & F_{M_1,M_2} is indeed a functor $A\text{-Mod}\to\text{Sets}$.

1.1) Definition of tensor product:

Definition: By the **tensor product** $M_1\otimes_A M_2$ we mean a representing object for $\mathrm{Bilin}_A(M_1\times M_2,\bullet)$ i.e. want a functor isomorphism $\mathrm{Hom}_A(M_1\otimes_A M_2,\bullet)\xRightarrow[\eta]{\sim}\mathrm{Bilin}_A(M_1\times M_2,\bullet)$

Similarly to products (Sec 2 of Lec 18) we can equivalently define tensor products via a universal property: this is an

A-module $M_1 \otimes_A M_2$ w. a bilinear map $M_1 \times M_2 \longrightarrow M_1 \otimes_A M_2$, $(m_1, m_2) \longmapsto m_1 \otimes m_2$, w. the following universal property:

(*) $\forall$ A-module N & A-bilinear map $\beta: M_1 \times M_2 \to N$ $\exists!$ A-linear map $\tilde{\beta}: M_1 \otimes_A M_2 \longrightarrow N$ s.t. $\beta(m_1, m_2) = \tilde{\beta}(m_1 \otimes m_2)$

i.e. the following is commutative

$$\begin{tikzcd} M_1 \times M_2 \arrow[d, "(m_1,m_2) \mapsto m_1 \otimes m_2"'] \arrow[dr, "\beta"] & \\ M_1 \otimes_A M_2 \arrow[r, dashed, "\tilde{\beta}"] & N \end{tikzcd}$$

In terms of η, this universal bilinear map is $\eta_{M_1 \otimes_A M_2}(1_{M_1 \otimes_A M_2})$, and (*) yieds η_N, inverse to $Bilin_A(M_1, M_2; N) \to Hom_A(M_1 \otimes_A M_2, N)$, $\beta \mapsto \tilde{\beta}$; cf the proof of Lemma in Sec 2 of Lec 18.

Thx to either of the equivalent definitions, the tensor product is unique (if exists) in the following sense: if $M_1 \otimes'_A M_2$ is another tensor product w. bilinear map $(m_1, m_2) \mapsto m_1 \otimes' m_2$, then $\exists!$ A-module isomorphism $\iota: M_1 \otimes_A M_2 \xrightarrow{\sim} M_1 \otimes'_A M_2 \mid \iota(m_1 \otimes m_2) = m_1 \otimes' m_2 \ \forall m_i \in M_i$. Indeed thx to (*) we have A-linear maps

$\iota: M_1 \otimes_A M_2 \rightleftarrows M_1 \otimes'_A M_2 : \iota'$. Then $\iota \circ \iota'(m_1 \otimes m_2) = m_1 \otimes m_2 \ \forall m_i \in M_i$ & by the uniqueness part of the property, $\iota \circ \iota' = id$. Similarly, $\iota' \circ \iota = id$.

2) Construction

The main goal of this section is to give a constructive proof of

Theorem: $M_1 \otimes_A M_2$ exists for all A-modules M_1, M_2.

2.1) Case 1: M_1 is free.

Lemma: For any set I, $A^{\oplus I}\otimes_A M \xrightarrow{\sim} M^{\oplus I}$ w.
$$(a_i)_{i\in I}\otimes m = (a_i m)_{i\in I} \quad \forall (a_i)\in A^{\oplus I},\ m\in M.$$

Proof:

As for A-linear maps from free modules, $\beta\in \mathrm{Bilin}_A(A^{\oplus I}, M; N)$ is uniquely determined by $\beta_i \in \mathrm{Hom}_A(M,N)$, $\beta_i(m):=\beta(e_i,m)$, via $\beta((a_i),m) = \sum_{i\in I} a_i\beta_i(m)$. Note that the r.h.s. is well-defined b/c $\{i\in I \mid a_i\neq 0\}$ is finite.

With this observation we'll give two slightly different proofs: using the definition & using the universal property (*)

Proof using the definition:

We have a bijection $\mathrm{Bilin}_A(A^{\oplus I}, M; N)\xrightarrow{\sim}\mathrm{Hom}_A(M,N)^{\times I}$, $\beta\mapsto(\beta(e_i,\cdot))$. On the other hand, by Sec 1.2 in Sec 4, have bijection $\mathrm{Hom}_A(M,N)^{\times I}\longrightarrow \mathrm{Hom}_A(M^{\oplus I}, N)$, $(\beta_i)\mapsto[(m_i)_{i\in I}\mapsto \sum_{i\in I}\beta_i(m_i)]$. Consider the composition,
$$\eta_N' : \mathrm{Bilin}_A(A^{\oplus I}, N)\xrightarrow{\sim}\mathrm{Hom}_A(M^{\oplus I}, N),\ \beta\mapsto[(m_i)\mapsto\sum\beta(e_i,m_i)].$$
These bijections constitute a functor isomorphism (exercise). This shows that $M^{\oplus I}$ is indeed $A^{\oplus I}\otimes_A M$ (take $\eta=(\eta')^{-1}$). Note that $\beta((a_i),m):=(a_i m)_{i\in I}$ is in $\mathrm{Bilin}(A^{\oplus I}, M; M^{\oplus I})$ & satisfies $\eta'(\beta) = 1_{M^{\oplus I}}$ (exercise) so indeed, $(a_i)\otimes m = (a_i m)$. $\square$

Proof using (*):

Let $\beta: A^{\oplus I}\times M\to N$ be a bilinear & let $\gamma: A^{\oplus I}\times M\to M^{\oplus I}$

be given by $((a_i)_{i\in I}, m) \mapsto (a_i m)_{i\in I}$, it's bilinear (exercise). We want to construct a linear map $\tilde{\beta}: M^{\oplus I} \to N$ s.t. $\beta = \tilde{\beta}\circ\gamma$ & show that the latter equation determines $\tilde{\beta}$ uniquely.

Construction: $\tilde{\beta}((m_i)_{i\in I}) = \sum_{i\in I} \beta_i(m_i)$ (where $\beta_i = \beta(e_i, \cdot)$).

We check $\beta = \tilde{\beta}\circ\gamma$: $\tilde{\beta}\circ\gamma((a_i), m) = \tilde{\beta}((a_i m)) = \sum \beta_i(a_i m) = \sum a_i \beta_i(m_i)$
$= \sum a_i \beta(e_i, m) = \beta(\sum a_i e_i, m) = \beta((a_i), m)$ ✓.

The uniqueness of $\tilde{\beta}$ satisfying $\beta = \tilde{\beta}\circ\gamma$ is an exercise. $\square$

2.2) Step 2: arbitrary M_1.

Let M_1', M_2 be A-modules s.t. $M_1'\otimes_A M_2$ exists. Let $K_1 \subset M_1'$ be an A-submodule $\rightsquigarrow M_1 := M_1'/K_1$ & $\pi_1: M_1' \twoheadrightarrow M_1$. Inside $M_1'\otimes_A M_2$ consider submodule $K := \mathrm{Span}_A(k_1\otimes m_2 \mid k_1\in K_1, m_2\in M_2)$.
$\rightsquigarrow$ projection $\pi: M_1'\otimes_A M_2 \twoheadrightarrow M_1'\otimes_A M_2/K$.

Claim: $M_1'\otimes_A M_2/K$ is the tensor product $M_1\otimes_A M_2$ & for $m_1 = \pi_1(m_1') \in M_1$ & $m_2\in M_2$, have $m_1\otimes m_2 := \pi(m_1'\otimes m_2)$.

How this implies Thm: M_1 is a quotient of $A^{\oplus I}$ for some I (if $(m_i)_{i\in I}$ are generators of M_1, then $\varphi_{\underline{m}}: A^{\oplus I} \to M_1$, $(a_i) \mapsto \sum_{i\in I} a_i m_i$, is surjective). $A^{\oplus I}\otimes_A M$ exists by Step 1. Now in Claim take $M_1' := A^{\oplus I}$ to see that $M_1\otimes_A M_2$ exists.

Proof of Claim.

Exercise: $m_1\otimes m_2$ is well-defined (independent of choice of

$m_1')$ & $(m_1,m_2)\mapsto m_1\otimes m_2$ is a bilinear map $M_1\times M_2 \to M_1'\otimes_A M_2/K$.
Now we only need to check univ'l property (II): $\forall$ bilinear $\beta: M_1\times M_2 \to N$ $\exists!$ linear $\tilde{\beta}: M_1'\otimes_A M_2/K \to N$ s.t. $\beta(m_1,m_2)=\tilde{\beta}(m_1\otimes m_2)$.

Define $\beta': M_1'\times M_2 \to N$ by $\beta'(m_1',m_2)=\beta(\pi_1(m_1'),m_2)$ so β' is bilinear $\rightsquigarrow$ $\exists!$ $\tilde{\beta}': M_1'\otimes_A M_2 \to N$ s.t. $\tilde{\beta}'(m_1'\otimes m_2) = \beta'(m_1',m_2)$. Note that $\tilde{\beta}'(k_1\otimes m_2)=\beta'(k_1,m_2)=\beta(0,m_2)=0$ so $\tilde{\beta}'(K)=0$. So $\exists!$ $\tilde{\beta}: M_1'\otimes_A M_2/K \to N$ s.t. $\tilde{\beta}'=\tilde{\beta}\circ\pi$. This is precisely the cond'n $\tilde{\beta}(m_1\otimes m_2)=\beta(m_1,m_2)$. The uniqueness of $\tilde{\beta}$ is an exercise. $\square$

2.3) Examples.

1) Tensor product of free modules: $A^{\oplus I}\otimes_A A^{\oplus J} = [\text{Step 1}] = (A^{\oplus J})^{\oplus I} \simeq A^{\oplus (I\times J)}$ w. basis $e_i\otimes e_j$ $(i\in I, j\in J)$.

An important observation is that not every element of $M_1\otimes_A M_2$ has the form $m_1\otimes m_2$ (we'll call such elements elementary tensors or tensor monomials.) Let's illustrate this by continuing the example:

Let $A=\mathbb{F}$ be a field, $M_1=\mathbb{F}^{\oplus k}$, $M_2=\mathbb{F}^{\oplus \ell}$. By Example 1, $M_1\otimes_{\mathbb{F}} M_2 \xrightarrow{\sim} \{\sum_{i=1}^{\ell}\sum_{j=1}^{\ell} a_{ij}\, e_i\otimes e_j\} \cong \{k\times\ell\text{-matrices}\}$.

Under this identification, the tensor monomials correspond to rk 1 matrices (exercise).

2) Let $I \subset A$ be an ideal and M be an arbitrary A-module. We claim that $(A/I) \otimes_A M \xrightarrow{\sim} M/IM$. Indeed, in Step 2, we take $M_1' = A$, $M_1 = A/I$, $M_2 = M$. By Step 1, $M_1' \otimes_A M_2 = A \otimes_A M \xrightarrow{\sim} M$ w. $a \otimes m \mapsto am$. Then $K_1 = I$ and $K = \mathrm{Span}_A(b \otimes m \mid b \in I, m \in M) \subset A \otimes_A M$. Under the isomorphism $A \otimes_A M \xrightarrow{\sim} M$, the submodule K corresponds to $\mathrm{Span}_A(bm \mid b \in I, m \in M) = IM$. So, by Step 2, $(A/I) \otimes_A M \xrightarrow{\sim} M/IM$.

2') As a concrete example of 2), take $A = \mathbb{C}[x]$, $I = (f)$ & $M = A/(g)$ for $f, g \in A$, so we are computing $\mathbb{C}[x]/(f) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(g)$. By 2), it's enough to compute $M/IM = [\mathbb{C}[x]/(g)]/[f(\mathbb{C}[x]/g)]$ $= \mathbb{C}[x]/(f,g) = \mathbb{C}[x]/(GCD(f,g))$.

In particular, if $GCD(f,g) = 1$, then $A/I \otimes_A M = \{0\}$ - so the tensor product of nonzero modules can be 0.

BONUS: Tensor products over noncommutative rings.

Let A be a comm've ring & R be an A-algebra (associative but perhaps non-commutative). Recall that it makes sense to talk about left & right R-modules & also about bimodules. Also (compare to Bonus of Lec 3) for two left R-modules M_1, M_2, the Hom set $\mathrm{Hom}_R(M_1, M_2)$ is only an A-module, not an R-module.

As for tensor products, we can tensor left R-modules w. right R-modules. Namely, let M be a left R-module & N be a right R-module. For an A-module L consider the set

$\mathrm{Bilin}_R(N\times M, L)$ consisting of all A-bilinear maps $\varphi: N\times M \to L$ s.t. in addition $\varphi(nr, m) = \varphi(n, rm)$ $\forall\, r\in R$, $n\in N$, $m\in M$.

Definition: $N\otimes_R M \in \mathrm{Ob}(A\text{-Mod})$ represents the functor $\mathrm{Bilin}_R(N\times M, \bullet): A\text{-Mod} \longrightarrow \mathrm{Sets}$.

Important exercise: If R is comm've, then this definition gives the same as the definition in Sec 1.1.

To construct $N\otimes_R M$ we can use the same construction as we did in the lecture. Alternatively, $N\otimes_R M$ is the quotient of $N\otimes_A M$ by the A-submodule

$$\mathrm{Span}_A(nr\otimes m - n\otimes rm \mid n\in N, m\in M, r\in R).$$

Now suppose we have 2 more A-algebras, S and T. Let N be an S-R-bimodule & M be an R-T-bimodule.

Important exercise: $\exists !$ S-T-bimodule str're on $N\otimes_R M$ s.t. $s(n\otimes m) = sn\otimes m$, $(n\otimes m)t = n\otimes mt$.