Lecture 21: Tensor products, II.

1) Further discussion of tensor products.

2) Tensor-Hom adjunction.

Ref: [AM], Section 2.7

1) Further discussion of tensor products.

Let A be a commutative ring & M_1, M_2 be A-modules. In Lec 20 we have defined their tensor product $M_1\otimes_A M_2$ together with a bilinear map $M_1\times M_2 \to M_1\otimes_A M_2, (m_1,m_2)\mapsto m_1\otimes m_2$, with the following universal property: $\forall$ A-bilinear map $\beta: M_1\times M_2\to N$ $\exists!$ A-linear map $\tilde\beta: M_1\otimes_A M_2\to N$ s.t. $\beta(m_1,m_2)=\tilde\beta(m_1\otimes m_2)$ $\forall m_i\in M_i$.

1.1) Generators

Lemma: If $M_k = Span_A(m_k^i \mid i\in I_k)$ $k=1,2$, then
$M_1\otimes_A M_2 = Span_A(m_1^i\otimes m_2^j \mid i\in I_1, j\in I_2)$.
In particular $M_1\otimes_A M_2 = Span_A(m_1\otimes m_2 \mid m_i\in M_i)$.

Proof: Let $N = M_1\otimes_A M_2/ Span_A(m_1^i\otimes m_2^j\mid i\in I_1, j\in I_2)$, and let $\pi: M_1\otimes_A M_2 \twoheadrightarrow N$ be the projection. Consider $\beta: M_1\times M_2\to N$, $(m_1,m_2)\mapsto \pi(m_1\otimes m_2)$, it's A-bilinear. We have $\beta(m_1^i,m_2^j)=0$ and since $M_1=Span_A(m_1^i)$, $M_2=Span_A(m_2^j)$, from bilinearity we get $\beta=0$. Now, π is the <u>unique</u> linear map w. $\pi(m_1\otimes m_2)=\beta(m_1,m_2)$ and since 0 also satisfies this equality, $\pi=0$. Since π is surjective, $N=0 \Leftrightarrow m_1^i\otimes m_2^j$ span $M_1\otimes_A M_2$ $\square$

1.2) Tensor products of linear maps.

Let M_1, M_1', M_2, M_2' be A-modules & $\varphi_i \in \mathrm{Hom}_A(M_i, M_i'), i=1,2$.
Consider: $M_1 \times M_2 \longrightarrow M_1' \otimes_A M_2'$, $(m_1, m_2) \mapsto \varphi_1(m_1) \otimes \varphi_2(m_2)$

Exercise: This map is A-bilinear.

So it gives rise to an A-linear map $\varphi_1 \otimes \varphi_2: M_1 \otimes_A M_2 \rightarrow M_1' \otimes_A M_2'$ uniquely characterized by $[\varphi_1 \otimes \varphi_2](m_1 \otimes m_2) = \varphi_1(m_1) \otimes \varphi_2(m_2)$ $\forall m_i \in M_i$.

Properties of tensor products of maps:

- $id_{M_1} \otimes id_{M_2} = id_{M_1 \otimes_A M_2}$
- Compositions: $M_1 \xrightarrow{\varphi_1} M_1' \xrightarrow{\varphi_1'} M_1''$, $M_2 \xrightarrow{\varphi_2} M_2' \xrightarrow{\varphi_2'} M_2''$
  $(\varphi_1' \varphi_1) \otimes (\varphi_2' \varphi_2) = (\varphi_1' \otimes \varphi_2')(\varphi_1 \otimes \varphi_2)$ b/c they coincide on generators $m_1 \otimes m_2$ of $M_1 \otimes_A M_2$.

So: we have the tensor product functor
$$A\text{-Mod} \times A\text{-Mod} \longrightarrow A\text{-Mod}$$
We will usually fix an A-module L & consider the functor $L \otimes_A \bullet : A\text{-Mod} \rightarrow A\text{-Mod}$ sending an A-module M to $L \otimes_A M$ & an A-linear map $\varphi: M \rightarrow M'$ to $id_L \otimes \varphi$.

Rem: Analogously to Case 2 in Sec 2.1 of Lec 20, if $K \subset M$ is a submodule, then $L \otimes_A (M/K)$ is the quotient of $L \otimes_A M$ by $\mathrm{Span}_A(l \otimes k \mid l \in L, k \in K)$. The latter submodule can be described as $\mathrm{im}(\mathrm{id}_L \otimes \iota)$, where $\iota: K \hookrightarrow M$ is the inclusion.

Important exercise: Prove that $(\varphi_1, \varphi_2) \mapsto \varphi_1 \otimes \varphi_2$:
$\mathrm{Hom}_A(M_1, M_1') \times \mathrm{Hom}_A(M_2, M_2') \longrightarrow \mathrm{Hom}_A(M_1 \otimes_A M_2, M_1' \otimes_A M_2')$
is A-bilinear (hint: check on generators of $M_1 \otimes_A M_2$)

1.3) "Algebra properties" of tensor products.

Theorem: Let M_1, M_2, M_3 be A-modules. Then:

1) There is a unique isomorphism $(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A (M_2 \otimes_A M_3)$ s.t. $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$. (i.e. tensor product is associative).
2) $\exists!$ isom'm $M_1 \otimes_A M_2 \xrightarrow{\sim} M_2 \otimes_A M_1$ w. $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$.
3) $\exists!$ isom'm $M_1 \otimes_A (M_2 \oplus M_3) \xrightarrow{\sim} M_1 \otimes_A M_2 \oplus M_1 \otimes_A M_3$ w. $m_1 \otimes (m_2, m_3) \mapsto (m_1 \otimes m_2, m_1 \otimes m_3)$
4) $\exists!$ unique isom'm $A \otimes_A M \xrightarrow{\sim} M$ s.t. $a \otimes m \mapsto am$.

Proof: (1)

We want to establish the existence of an A-linear map
$\tilde{\beta}: (M_1 \otimes_A M_2) \otimes_A M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$ s.t. $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$
Such a map will be unique b/c the elements $m_1 \otimes m_2$ span $M_1 \otimes_A M_2$ hence $(m_1 \otimes m_2) \otimes m_3$ span $(M_1 \otimes_A M_2) \otimes_A M_3$, see Sec 1.1.
So, we need a bilinear map $\beta: (M_1 \otimes_A M_2) \times M_3 \longrightarrow M_1 \otimes_A (M_2 \otimes_A M_3)$ s.t. $(m_1 \otimes m_2, m_3) \mapsto m_1 \otimes (m_2 \otimes m_3)$.
Fix $m_3 \rightsquigarrow$ a linear map $M_2 \rightarrow M_2 \otimes_A M_3$, $m_2 \mapsto m_2 \otimes m_3$. Define

$\beta_{m_3}: M_1\otimes_A M_2 \longrightarrow M_1\otimes_A (M_2\otimes_A M_3)$ to be the tensor product of id_{M_1} & $[m_2\mapsto m_2\otimes m_3]$ so $\beta_{m_3}(m_1\otimes m_2) = m_1\otimes(m_2\otimes m_3)$

Note that β_{m_3} depends linearly on m_3 (e.g. $\beta_{am_3}=a\beta_{m_3}$ b/c both send $m_1\otimes m_2$ to $a\cdot m_1\otimes(m_2\otimes m_3)$ & $\mathrm{Span}_A(m_1\otimes m_2)=M_1\otimes_A M_2$).

$\rightsquigarrow$ A-bilinear map $\beta: (M_1\otimes_A M_2)\times M_3 \longrightarrow M_1\otimes_A(M_2\otimes_A M_3)$, $\beta(x,m_3):=\beta_{m_3}(x) \rightsquigarrow \tilde{\beta}$ as needed.

$\tilde{\beta}$ is an isomorphism: have $\tilde{\beta}': M_1\otimes_A(M_2\otimes_A M_3)\longrightarrow (M_1\otimes_A M_2)\otimes M_3$ $m_1\otimes(m_2\otimes m_3)\mapsto (m_1\otimes m_2)\otimes m_3$. It's inverse of $\tilde{\beta}$ b/c $\tilde{\beta}'\circ\tilde{\beta}=id$ & $\tilde{\beta}\circ\tilde{\beta}'=id$ on generators $(m_1\otimes m_2)\otimes m_3$. $\square$ of (1).

(2) - commutativity - is an exercise & (4) - unit - follows from our construction (Case 1 in Sec 2.1 of Lec 20)

Proof of (3) - distributivity: consider the projection $\pi_i: M_2\oplus M_3 \longrightarrow M_i$, $i=2,3$; & inclusion $\iota_i: M_i \hookrightarrow M_2\oplus M_3$

$\rightsquigarrow id_{M_1}\otimes\pi_i: M_1\otimes_A(M_2\oplus M_3) \rightleftarrows M_1\otimes_A M_i: id_{M_1}\otimes\iota_i \rightsquigarrow$

$$(id_{M_1}\otimes\pi_2, id_{M_1}\otimes\pi_3): M_1\otimes_A(M_2\oplus M_3) \rightleftarrows M_1\otimes_A M_2\oplus M_1\otimes_A M_3 : (id_{M_1}\otimes\iota_2, id_{M_1}\otimes\iota_3)$$

$$id_{M_1}\otimes\iota_2(x)+id_{M_1}\otimes\iota_3(y) \longleftarrow\!\shortmid (x,y)$$

Exercise: check that these maps are mutually inverse. $\square$

2) Tensor-Hom adjunction.

The goal of this section is to prove that tensor product functors are left adjoint to Hom functors.

2.1) Basic setting.

Let L be an A-module. We can consider the following functors $A\text{-Mod} \to A\text{-Mod}$:

1) $L \otimes_A \bullet$ from Sec 1.2.
2) $\underline{\mathrm{Hom}}_A(L,\cdot)$ defined exactly as $\mathrm{Hom}_A(L,\cdot): A\text{-Mod} \to \text{Sets}$ but viewed as a functor to $A\text{-Mod}$, which makes sense b/c for an A-linear map $\psi: M \to M'$, the map $\psi \circ ?: \mathrm{Hom}_A(L,M) \to \mathrm{Hom}_A(L,M')$ is A-linear (Prob 4 in HW1). Formally, $\mathrm{Hom}_A(L,\cdot) \xrightarrow{\sim} \mathrm{For} \circ \underline{\mathrm{Hom}}_A(L,\bullet)$, where For is the forgetful functor $A\text{-Mod} \to \text{Sets}$.

Thm (tensor-Hom adjunction): $L \otimes_A \bullet$ is left adjoint to $\underline{\mathrm{Hom}}_A(L,\cdot)$ (as functors $A\text{-Mod} \to A\text{-Mod}$).

Proof: We need to construct "natural" bijections of sets $\eta_{M,N}: \mathrm{Hom}_A(L\otimes_A M, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_A(L,N))$ (M,N are A-modules) & check the commutativity of two diagrams. Pick $\tau \in \mathrm{Hom}_A(L\otimes_A M, N)$. Want to get $\varphi_\tau \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(L,N))$. Choose $m \in M$. Then $\ell \mapsto \ell\otimes m: L \to L\otimes_A M$ is a linear map, hence $\tau_m := \ell \mapsto \tau(\ell\otimes m): L \to N$ is a linear map, i.e. an element of $\mathrm{Hom}_A(L,N)$. So we get a map $\varphi^\tau: M \to \mathrm{Hom}_A(L,N)$, $m \mapsto \tau_m$. It's A-linear: e.g. for $a\in A$, $\tau_{am}(\ell) = \tau(\ell\otimes am) = [(\ell,m)\mapsto \ell\otimes m$ is A-bilinear$] = a\tau(\ell\otimes m) = [a\tau_m](\ell)$ $\forall \ell \in L$. So $\varphi^\tau \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(L,N))$. We set $\eta_{M,N}(\tau) = \varphi^\tau$.

Conversely, let $\varphi \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(L,N))$. Then $(\ell,m) \mapsto [\varphi(m)](\ell)$ is A-bilinear map $L\times M \to N$ (exercise). Let τ_φ be the corresponding

A-linear map $L\otimes_A M\to N$, uniquely characterized by

$$\tau_\varphi(\ell\otimes m)=[\varphi(m)](\ell)$$

The maps $\tau\mapsto\varphi^\tau$ & $\varphi\mapsto\tau_\varphi$ are inverse to each other: e.g. let's check $\tau_{\varphi^\tau}=\tau$. Since $L\otimes_A M$ is spanned by $\ell\otimes m$ w. $\ell\in L, m\in M$, it's enough to check the equality on these elements:

$$\tau_{\varphi^\tau}(\ell\otimes m)=[\varphi_\tau(m)](\ell)=\tau_m(\ell)=\tau(\ell\otimes m) \qquad \checkmark$$

Let's check that the bijections η_{MN}'s make one diagram in the definition of adjoint functors (Sec 2.1 of Lec 19) commutative (the other is an exercise). Pick $\zeta\in Hom_A(M,M')$. We need to show the following is commutative

$$\begin{tikzcd}
Hom_B(L\otimes_A M',N) \arrow[r, "\eta_{M',N}"] \arrow[d, "?\circ(id_L\otimes\zeta)"] & Hom_A(M', Hom_A(L,N)) \arrow[d, "?\circ\zeta"] \\
Hom_B(L\otimes_A M,N) \arrow[r, "\eta_{M,N}"] & Hom_A(M, Hom_A(L,N))
\end{tikzcd}$$

$\downarrow\longrightarrow$: $\tau\mapsto[m\mapsto[\ell\mapsto\tau\circ(id_L\otimes\zeta)(\ell\otimes m)=\tau(\ell\otimes\zeta(m))]]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\| \qquad\qquad \checkmark$

$\longrightarrow\downarrow$: $\tau\mapsto[m\mapsto[\eta_{M',N}(\tau)](\zeta(m))=\tau(\ell\otimes\zeta(m))]$ $\qquad\square$

2.2) Generalization.

It turns out that the same method gives left (and right) adjoint functors to pullback functors $\varphi^*: B\text{-Mod}\to A\text{-Mod}$ (Sec 1.2 of Lec 17) for $\varphi: A\to B$, a homomorphism of commutative rings. These adjoints are important so we explore a more general setup.

Let L be a B-module (so also an A-module) & M be an A-module $\rightsquigarrow$ A-module $L \otimes_A M$.

Lemma: 1) There is a unique B-module structure on $L \otimes_A M$ s.t.
$$b(l \otimes m) = (bl) \otimes m \quad \forall\, b \in B,\ l \in L,\ m \in M$$
2) If $\varphi: M \to M'$ is an A-linear map, then $id_L \otimes \varphi$ is a B-linear map $L \otimes_A M \to L \otimes_A M'$.

Proof: 1) Consider the map $\beta_b: L \times M \to L \otimes_A M$, $(l, m) \mapsto (bl) \otimes m$. It's A-bilinear (exercise) so $\exists!$ A-linear map $\tilde{\beta}_b: L \otimes_A M \to L \otimes_A M$ s.t. $\tilde{\beta}_b(l \otimes m) = (bl) \otimes m$ ($\forall\, b \in B, l \in L, m \in M$). Define a map
$$B \times (L \otimes_A M) \longrightarrow L \otimes_A M, \quad (b, x) \mapsto \tilde{\beta}_b(x).$$
We claim that it defines a B-module structure on $L \otimes_A M$. This is a boring check of axioms using that $\tilde{\beta}_b$ is A-linear & $\mathrm{Span}_A(l \otimes m) = L \otimes_A M$ (Sec 1.1) For example, to check associativity, $(b_1 b_2)x = b_1(b_2 x)$ it's enough to assume that $x = l \otimes m$. Then $(b_1 b_2)x = (b_1 b_2 l) \otimes m$
$= b_1(b_2(l \otimes m)) = b_1(b_2 x)$.

2) is left as an exercise. $\square$

This lemma gives us a <u>functor</u> $L \otimes_A \cdot : A\text{-Mod} \to B\text{-Mod}$.