Lecture 22: tensor products, III.

1) Tensor-Hom adjunction, cont'd.
2) Tensor products of algebras.

Refs: [AM], Secs 2.8, 2.11
BONUS: Induction of group representations.

1) Tensor-Hom adjunction, cont'd.

1.1) Generalization of tensor-Hom adjunction

This is a continuation of Sec 2.2 in Lec 21. Let $\varphi: A \to B$ be a homomorphism of commutative rings. This gives rise to the forgetful functor $\varphi^*: B\text{-Mod} \longrightarrow A\text{-Mod}$.

Now let L be a B-module. In Sec 2.2 of Lec 21, we interpreted $L\otimes_A ?$ as a functor $A\text{-Mod} \to B\text{-Mod}$ (the B-action on $L\otimes_A M$ is uniquely determined by $b(\ell\otimes m) = (b\ell)\otimes m$). On the other hand, we have a functor $\varphi^* \underline{\mathrm{Hom}}_B(L,\cdot): B\text{-Mod} \to A\text{-Mod}$.

Thm (Tensor-Hom adjunction): The functor $L\otimes_A \cdot : A\text{-Mod} \to B\text{-Mod}$ is left adjoint to $\varphi^*\underline{\mathrm{Hom}}_B(L,\cdot): B\text{-Mod} \to A\text{-Mod}$.

Sketch of proof: the proof closely follows that in Sec 2.1, Lec. 21. We need to show $\tau_m: L \to N$ & $\tau_\varphi: L\otimes_A M \to N$ are B-linear. Both claims follow directly from the construction of the B-action on $L\otimes_A M$. $\square$

1.2) Base change

Take $L=B$. Then $\underline{\mathrm{Hom}}_B(B,N)$ is naturally isomorphic to N for any B-module N, i.e. $\underline{\mathrm{Hom}}_B(B,\cdot)$ is isomorphic to the identity endo-functor of B-Mod. We arrive at the following:

Corollary: The functor $B\otimes_A\cdot: A\text{-Mod}\to B\text{-Mod}$ (base change or induction functor) is left-adjoint to $\varphi^*: B\text{-Mod}\to A\text{-Mod}$.

We encounter base change first when in Linear algebra we replace vector spaces over a field $\mathbb{F}$ w. vector spaces over the algebraic closure $\overline{\mathbb{F}}$ (e.g. $\mathbb{F}=\mathbb{R}$, $\overline{\mathbb{F}}=\mathbb{C}$), this is basically done by applying $\overline{\mathbb{F}}\otimes_{\mathbb{F}}\cdot$.

Here's another appearance of base change from this course.

Proposition: Let $S\subset A$ be a multiplicative subset. Then the functor $A[S^{-1}]\otimes_A\cdot: A\text{-Mod}\to A[S^{-1}]\text{-Mod}$ is isomorphic to the localization functor $\cdot[S^{-1}]$.

Proof:

By Example 2 in Sec 2.2 in Lec 19, $\cdot[S^{-1}]$ is left adjoint to the pullback functor $A[S^{-1}]\text{-Mod}\to A\text{-Mod}$. By Corollary above, so is $A[S^{-1}]\otimes_A\cdot$. Now the uniqueness of adjoints (Sec 2.3 of Lec 19) guarantees $\cdot[S^{-1}]\xrightarrow{\sim}A[S^{-1}]\otimes_A\cdot$ $\square$

Rem: Here's a concrete way to think about $B\otimes_A M$ (under mild assumptions on M). Namely assume $\exists\, k,\ell\in\mathbb{Z}_{\geqslant 0}$ & A-linear map $\tau: A^{\oplus \ell}\longrightarrow A^{\oplus k}$ s.t. $M \xrightarrow{\sim} A^{\oplus k}/\operatorname{im}\tau$. Then τ is given by a matrix $T=(a_{ij})\in \operatorname{Mat}_{k\times \ell}(A)$ (we view elements of $A^{\oplus k}, A^{\oplus \ell}$ as column vectors). This is a way to present M by generators & relations

We write $\varphi(T)$ for the element $(\varphi(a_{ij}))\in \operatorname{Mat}_{k\times\ell}(B)$ & let ${}^{\varphi}\tau$ be the corresponding B-linear map $B^{\oplus k}\longrightarrow B^{\oplus \ell}$. The following generalizes Sec 1.3 in Lec 10.

Exercise: $B\otimes_A (A^{\oplus k}/\operatorname{im}\tau) \xrightarrow{\sim} B^{\oplus k}/\operatorname{im}{}^{\varphi}\tau$.

2) Tensor product of algebras.

2.1) Construction.

Let A be a commutative ring, B, C be A-algebras (& so A-modules) $\rightsquigarrow$ A-module $B\otimes_A C$.

Proposition: $\exists!$ A-algebra structure on $B\otimes_A C$ s.t. $(b_1\otimes c_1)\cdot(b_2\otimes c_2) = b_1b_2\otimes c_1c_2$ $\forall\, b_1,b_2\in B,\ c_1,c_2\in C$ (w. unit $1\otimes 1$).

Proof: Uniqueness will follow b/c $B\otimes_A C = \operatorname{Span}_A(b\otimes c \mid b\in B, c\in C)$ (Sec 1.1 of Lec 21) & any bilinear map is uniquely determined by images of generators.

Now we need to show the existence. The product map $B\times B\longrightarrow B$ is A-bilinear $\rightsquigarrow \exists!$ A-linear

$$\mu_B: B\otimes_A B\longrightarrow B, \text{ w. } b_1\otimes b_2\mapsto b_1b_2.$$

Similarly, we have $\mu_C: C\otimes_A C\longrightarrow C \rightsquigarrow$

$$\begin{tikzcd}
(B\otimes_A B)\otimes_A(C\otimes_A C) \arrow[rr, "\mu_B\otimes\mu_C"] & & B\otimes_A C \\
x\otimes y \in (B\otimes_A C)\otimes_A(B\otimes_A C) \arrow[u, "\wr"] \arrow[urr] & & \\
(x,y)\in(B\otimes_A C)\times(B\otimes_A C) \arrow[u] \arrow[uurr] & &
\end{tikzcd}$$

assoc. & commut. of $\otimes$ Sec 1.2 of Lec 16

$(b_1\otimes c_1)\otimes(b_2\otimes c_2)\mapsto(b_1b_2)\otimes(c_1c_2)$

our multiplication map

So we've shown existence of A-bilinear product map. Associativity & unit axioms can be checked on elementary tensor, e.g. here is a part of unit axiom:
$(1\otimes 1)(b\otimes c)=(1\otimes b)\otimes(1\otimes c)=b\otimes c.$ $\square$

Rem: B, C are commutative $\Rightarrow$ so is $B\otimes_A C$.

2.2) Coproduct.

Theorem: Let B, C be commutative. Then $B\otimes_A C$ is the coproduct of B&C in $\mathcal{E}:=A\text{-}CommAlg$ (the category of commutative A-algebras).

Recall (Sec 1 of Lec 19) that this means that the following functors are isomorphic
$$Hom_{\mathcal{E}}(B\otimes_A C,\cdot),\ Hom_{\mathcal{E}}(B,\cdot)\times Hom_{\mathcal{E}}(C,\cdot):\mathcal{E}\to Sets.$$
Equivalently:
$\exists$ A-algebra homomorphisms $\iota^B: B\to B\otimes_A C$, $\iota^C: C\to B\otimes_A C$ s.t. $\forall$ alg. homom's $\varphi^B: B\to D$, $\varphi^C: C\to D$, where D is

a commutative A-algebra, $\exists!$ A-alg. homom. φ:
w. $\varphi^B = \varphi \circ \iota^B (: B \to \mathcal{D})$ & $\varphi^C = \varphi \circ \iota^C (: C \to \mathcal{D})$.

Proof: Construction of ι^B, ι^C:

$\iota^B(b) := b \otimes 1$, $\iota^C(c) := 1 \otimes c$.

The conditions on $\varphi: B \otimes_A C \to \mathcal{D}$ we need to achieve:

$$\varphi(b\otimes 1) = \varphi^B(b),\ \varphi(1\otimes c) = \varphi^C(c) \Leftrightarrow [b\otimes c = (b\otimes 1)(1\otimes c)]$$

$$(*) \qquad \varphi(b\otimes c) = \varphi^B(b)\varphi^C(c).$$

We need to show $\exists!$ A-algebra homom'm $\varphi: B\otimes_A C \to \mathcal{D}$ satisfying $(*)$. The map $B\times C \to \mathcal{D}$, $(b,c) \mapsto \varphi^B(b)\varphi^C(c)$ is A-bilinear, so $\exists!$ <u>A-linear</u> φ satisfying $(*)$.

What remains to check is: φ respects ring multiplication (unit is clear) enough to do this on elementary tensors

$$\varphi(b_1\otimes c_1 \cdot b_2\otimes c_2) = \varphi(b_1b_2 \otimes c_1c_2) = \varphi^B(b_1b_2)\varphi^C(c_1c_2) =$$
$$= \varphi^B(b_1)\varphi^B(b_2)\varphi^C(c_1)\varphi^C(c_2) = [\mathcal{D} \text{ is commutative}] = (\varphi^B(b_1)\varphi^C(c_1))\cdot$$
$$(\varphi^B(b_2)\varphi^C(c_2)) = \varphi(b_1\otimes c_1)\varphi(b_2\otimes c_2) \qquad \square$$

Example: $B = A[x_1,\dots,x_k]/(f_1,\dots f_{k'})$, $C = A[y_1,\dots y_\ell]/(g_1,\dots g_{\ell'})$.
Then $B\otimes_A C \simeq A[x_1,\dots x_k, y_1,\dots y_\ell]/(\underbrace{f_1,\dots f_{k'}}_{\text{on } x_1\dots x_k}, \underbrace{g_1,\dots,g_{\ell'}}_{\text{on } y_1\dots y_\ell})$, denote the right hand side by $\mathcal{D}$.

Will show isomorphism of functors: $F_{\mathcal{D}} \xrightarrow{\sim} F_B \times F_C$ (where $F_{\mathcal{D}} = \mathrm{Hom}_{\mathcal{E}}(\mathcal{D},\cdot)$: $\mathcal{E}\to$ Sets & F_B, F_C are defined similarly), then we are done by the uniqueness of representing object, Sec 1.3 of Lec 18.

Define another functor $F'_B: \mathcal{E} \to Sets$ sending a comm've A-algebra R to $\{(r_1,\dots,r_k) \in R^k \mid f_i(r_1,\dots,r_k)=0, i=1,\dots k'\}$ and an A-algebra homomorphism $\psi: R^1 \to R^2$ to $F'_B(\psi): F'_B(R^1) \to F'_B(R^2)$, $(r_1,\dots,r_k) \mapsto (\psi(r_1),\dots,\psi(r_k))$.
-well-defined map b/c $f_i(\psi(r_1),\dots\psi(r_k)) = \psi(f_i(r_1,\dots r_k)) = 0$.

Then $F_B \overset{\sim}{\Rightarrow} F'_B$: $\varphi \in Hom_{A\text{-}Alg}(B,R)$ is sent to $(\varphi(\bar{x}_1),\dots,\varphi(\bar{x}_r)) \in R^k$ here $\bar{x}_i$ = image of x_i in B; $(\varphi(\bar{x}_1),\dots,\varphi(\bar{x}_k)) \in F'_B(R)$ similarly to the above. the map $\eta_R: \varphi \mapsto (\varphi(\bar{x}_1),\dots,\varphi(\bar{x}_r))$ is a bijection (by the description of homomorphisms from algebras given by generators & relations, Exercise 2 in Sec 0 of Lec 2). To show (η_R) constitute a functor (iso)morphism is an exercise.

Similarly, we have $F_C \overset{\sim}{\Rightarrow} F'_C$, $F_D \overset{\sim}{\Rightarrow} F'_D$. That $F'_D \overset{\sim}{\Rightarrow} F'_B \times F'_C$ is an exercise. This completes the example.

Concrete example: Take $A = \mathbb{C}[x]$, $B = \mathbb{C}[x]/(f)$, $C = \mathbb{C}[x]/(g)$. Here $k = \ell = 0$ (f, g are elements of A), so $B \otimes_A C = \mathbb{C}[x]/(f,g) = \mathbb{C}[x]/(GCD(f,g))$. Cf. Example 2') in Sec 2.3 of Lec 20.

Exercise: Let g_i^B be the image of $g_i \in A[x_1,\dots x_\ell]$ in $B[x_1,\dots x_\ell]$. Note the $B \otimes_A C$ is a B-algebra via c^B. Show that

$$B \otimes_A C \simeq B[x_1,\dots x_\ell]/(g_1^B,\dots,g_{\ell'}^B)$$

Bonus: induction of group representations.

This bonus is aimed at students who took Math 353 in (or know relevant representation theory). It's also based on Bonuses to Lecs 3 and 20.

Let A, B be general (associative unital) rings & $\varphi: A \to B$ be a homomorphism. Then it still makes sense to consider functor $B \otimes_A \bullet : A\text{-Mod} \longrightarrow B\text{-Mod}$

An interesting situation is as follows. Let $H \subset G$ be finite groups. Let $\mathbb{F}$ be a field. Set $A = \mathbb{F}H$, $B = \mathbb{F}G$ and let φ be the inclusion $A \hookrightarrow B$. The resulting functor is known as the induction of group representations. The claim that it's adjoint to the pullback functor (a.k.a. the restriction functor) is known as the Frobenius reciprocity.