Lecture 23: Exactness, projective & flat modules, I.

1) Additive functors

2) Exactness.

Refs: [AM], Secs 2.6, 2.9

BONUS: Additive & abelian categories.

0) What's next?

We will touch upon Homological algebra, a part of Algebra heavily inspired by (Algebraic) Topology. We will axiomatize some properties of functors $L\otimes_A ?$ & $\underline{Hom}_A(L,\cdot)$ to arrive at the notion of "additive functors" between categories of modules. The main question is the behavior of these functors on "exact sequences" of modules, which ultimately has to do with computing the images of objects under these functors. We'll see that $\underline{Hom}_A(L,\cdot)$ behaves best when L is "projective" & $L\otimes_A\cdot$ behaves best when L is "flat." These are two classes of modules of great importance throughout Algebra. We'll study projective modules in some level of detail.

1) Additive functors

1.1) Definition. Let A,B be commutative rings so that we can consider their categories of modules A-Mod, B-Mod. Hom sets in these categories are abelian groups.

Definition: A functor $F: A\text{-Mod} \to B\text{-Mod}$ is additive if $\forall$ A-modules M, N, the map $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_B(F(M), F(N))$, $\varphi \mapsto F(\varphi)$, is a group homomorphism.

Similarly, we can talk about additive functors $A\text{-Mod}^{opp} \to B\text{-Mod}$, here we require $\varphi \mapsto F(\varphi): \mathrm{Hom}_A(N,M) \to \mathrm{Hom}_B(F(M), F(N))$ to be a group homomorphism.

1.2) Examples.

0) Let $\varphi: A \to B$ be a homomorphism of commutative rings. The pullback functor $\varphi^*: B\text{-Mod} \to A\text{-Mod}$ is additive.

1) In the setting of 0), let L be a B-module. The functor $L \otimes_A \bullet : A\text{-Mod} \to B\text{-Mod}$ is additive (by Exercise in Sec 1.2 of Lec 21, the map $\varphi \mapsto id_L \otimes \varphi: \mathrm{Hom}_A(M, M') \to \mathrm{Hom}_B(L \otimes_A M, L \otimes_A M')$ is additive). In particular, thx to Proposition in Sec 1.2 of Lec 22, the localization functor $\bullet[S^{-1}] \xrightarrow{\sim} A[S^{-1}] \otimes_A \bullet$, so $\bullet[S^{-1}]$ is additive.

2) For an A-module M, the functor $\mathrm{Hom}_A(M, \cdot): A\text{-Mod} \to A\text{-Mod}$ is additive, see a) of Prob 4 of HW1.

2^{opp}) For an A-module N, the functor $\mathrm{Hom}_A(\cdot, N): A\text{-Mod}^{opp} \to A\text{-Mod}$ is additive, also a) Prob 4 of HW1.

3) Functor $\bullet^{\otimes 2}: A\text{-Mod} \to A\text{-Mod}$, $M^{\otimes 2} := M \otimes_A M$, $\varphi^{\otimes 2} = \varphi \otimes \varphi$ is not additive (exercise).

Side remark: There are more examples:

i) Tor & Ext functors that generalize tensor product & Hom functors.

ii) The homology & cohomology functors $H_k(X, \bullet)$ & $H^k(X, \bullet)$: $\mathbb{Z}\text{-Mod} \to \mathbb{Z}\text{-Mod}$, where X is a topological space. These are studied in Algebraic topology.

2) Exactness

This is the main property of additive functors we care about in Comm. algebra. It describes how a functor behaves on "exact sequences."

2.1) Exact sequences:

Let $M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{k-1}} M_k$ be a sequence of A-modules & their homomorphisms $\varphi_i \in \mathrm{Hom}_A(M_i, M_{i+1})$, $i = 0, \dots k-1$.

Definition: • this sequence is exact if $\operatorname{im} \varphi_{i-1} = \ker \varphi_i$ $\forall i = 1, \dots k-1$.

• A short exact sequence (SES) is an exact sequence of the form:

$$0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \to 0$$

i.e. φ_1 is injective, $\operatorname{im} \varphi_1 = \ker \varphi_2$ & φ_2 is surjective.

Example (of SES) if $N \subset M$ is an A-submodule, then have SES $0 \to N \to M \longrightarrow M/N \to 0$, where the 1st map is the inclusion, and the 2nd map is the projection.

In a way, every SES looks like in this example: φ_1 identifies M_1 w. submodule of M_2, φ_2 identifies M_3 w. $M_2/\operatorname{im}\varphi_1$.

2.2) Definition of exactness of functors.

Let A, B be commutative rings, $F: A\text{-Mod} \to B\text{-Mod}$ be an additive functor.

Definition (of left & right exact functors):

(i) If $\forall$ SES $0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \to 0$

the sequence $0 \to F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2)} F(M_3)$ is exact, then say F is left exact.

(ii) If $\forall$ SES as in (i), the sequence $F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3) \to 0$ is exact, then say F is right exact.

Rem: can define left/right exact functors $F: A\text{-Mod}^{opp} \to B\text{-Mod}$ e.g. in (i) require that

$$0 \to F(M_3) \xrightarrow{F(\varphi_2)} F(M_2) \xrightarrow{F(\varphi_1)} F(M_1)$$

is exact.

Def: For $F: A\text{-Mod} \to B\text{-Mod}$, or $A\text{-Mod}^{opp} \to B\text{-Mod}$, **exact** = left & right exact, i.e. sends SES to SES.

2.3) Examples:

0) For a ring homomorphism $\varphi: A \to B$, the pullback functor $\varphi^*: B\text{-Mod} \longrightarrow A\text{-Mod}$ is manifestly exact.

1) The tensor product functor $L\otimes_A \bullet\ A\text{-Mod} \to A\text{-Mod}$ is right exact: for a submodule $K \subset N$, $L\otimes_A (N/K)$ is the quotient of $L\otimes_A N$ by the image of $L\otimes_A K$, i.e the sequence $L\otimes_A K \to L\otimes_A N \to L\otimes_A (N/K) \to 0$ is exact, cf. Remark in Sec 1.2 of Lec 21. For general L, this functor is not exact, see Problem 4 in HW 4.

The same is true for $L\otimes_A \bullet: A\text{-Mod} \to B\text{-Mod}$ (the same as $L\otimes_A \bullet: A\text{-Mod} \to A\text{-Mod}$ on the level of abelian groups).

2) The localization functor $\bullet[S^{-1}]$ is exact: by Proposition in Section 1.1 of Lec 11, as it sends kernels to kernels & images to images, so SES to SES.

3) Let N be an A-module. Then $\underline{Hom}_A(\cdot, N): A\text{-Mod}^{opp} \to A\text{-Mod}$ is left exact, this follows from b) & c) of Problem 4 in HW1.

3^{opp}) $\underline{Hom}_A(M, \bullet): A\text{-Mod} \to A\text{-Mod}$ is left exact. Indeed, thanks to Example in Sec 2.1, it's enough to check that $\forall$ submodules

$K \subset N$, the sequence $0 \to \mathrm{Hom}_A(M,K) \xrightarrow{\iota\circ ?} \mathrm{Hom}_A(M,N) \xrightarrow{\pi\circ ?} \mathrm{Hom}_A(M,N/K)$, is exact, where $\iota: K \hookrightarrow N$ is the inclusion & $\pi: N \to N/K$ is the projection. Then $\ker[\pi\circ ?] = \mathrm{im}[\iota\circ ?]$ easily follows from $\pi\circ\varphi = 0 \Leftrightarrow \mathrm{im}\,\varphi \subset K$. Also $\iota\circ ?$ is injective b/c ι is.

Remark: Exactness properties give some ways to compute what functors do to objects, cf. Prob 4 in HW1 for Hom or Prob 4 in HW4 for $\otimes$. Exact functors are best for computations.

2.4) Consequences of definition.

Lemma: Let $F: A\text{-Mod} \to B\text{-Mod}$ be a left exact additive functor. Then

(a) F sends injections to injections.

(b) F sends every exact sequence $0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3$ to an exact sequence $0 \to F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3)$

(c) F is exact $\Leftrightarrow$ F sends surjections to surjections.

Proof: (a) $N \overset{\varphi_1}{\hookrightarrow} M$ can be included into SES

$$0 \to N \xrightarrow{\varphi_1} M \longrightarrow M' \longrightarrow 0, \quad M' := M/\mathrm{im}\,\varphi_1.$$

$$\big\downarrow F$$

$$0 \to F(N) \xrightarrow{F(\varphi_1)} F(M) \longrightarrow F(M') \text{ -exact} \Rightarrow F(\varphi_1) \text{ is injective.}$$

(b): $M_3' := \mathrm{im}\,\varphi_2 \subset M_3$; $\varphi_2' := \varphi_2$ viewed as a map to its image $\iota: M_3' \hookrightarrow M_3$: inclusion, so $\varphi_2 = \iota\circ\varphi_2'$.

$0 \to M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2'} M_3' \to 0$ is exact $\Rightarrow$

$$0 \to F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2')} F(M_3') \qquad (*)$$

is exact. Further, ι is injective $\Rightarrow$ [by (a)] $F(\iota)$ is injective

F is a functor $\Rightarrow F(\varphi_2) = F(\iota) \circ F(\varphi_2')$. So $\ker F(\varphi_2) = \ker F(\varphi_2')$.

By this and $(*)$, $0 \to F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3)$ is exact.

(c) is exercise. □

Rem: There are direct analogs of this lemma for <u>all other</u> types of one-sided exactness. E.g. left exact functor $F: A\text{-Mod}^{opp} \longrightarrow B\text{-Mod}$ sends $\forall$ exact sequence

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \text{ to exact sequence}$$

$$0 \to F(M_3) \longrightarrow F(M_2) \longrightarrow F(M_1) \quad \text{(exercise)}.$$

BONUS:

1) Additive categories.

In our definition of additive functors we need to consider categories A-Mod, A-Mod^{opp} separately. This is awkward. The concept of an "additive category" includes these examples & much more. And we can talk about additive functors between additive categories.

Definition: An additive category $\mathcal{C}$ is

(Data) · a category

· together w. abelian group structure on $Hom_{\mathcal{C}}(X,Y)$ $\forall\, X,Y \in Ob(\mathcal{C})$

These data have to satisfy the following axioms:

- $\exists\, 0 \in Ob(\mathcal{C})$ w. $Hom_{\mathcal{C}}(X,0) = Hom_{\mathcal{C}}(0,X) = \{0\}$.
- $\forall\, X,Y \in Ob(\mathcal{C})$, $\exists$ a product $X\times Y \in Ob(\mathcal{C})$.
- the composition map $Hom_{\mathcal{C}}(X,Y)\times Hom_{\mathcal{C}}(Y,Z) \to Hom_{\mathcal{C}}(X,Z)$

is bi-additive (a.k.a. $\mathbb{Z}$-bilinear), $\forall\, X,Y,Z \in Ob(\mathcal{C})$.

Recall that in $\mathbb{Z}$-Mod, the product of two objects (in fact, of any finite collection) coincides w. their coproduct. This property carries over to arbitrary additive categories. The (co)product $X\times Y$ is usually called the direct sum and is denoted by $X\oplus Y$.

Examples (of additive categories):

1) A-Mod (for a ring A, not necessarily comm've).

2) A-Mod^{opp}.

3) A full subcategory in an additive category is additive iff it's closed under taking finite direct sums. For example, in A-Mod we can consider the full subcategories consisting of free objects. They are closed under direct sums hence additive.

4*) In various parts of Geometry / Topology people consider categories of "sheaves". These categories are additive.

5*) Various constructions in Homological Algebra produce more complicated additive categories from A-Mod: homotopy categories of complexes, derived categories etc.

2) Abelian categories.

Additive functors make sense between additive categories. Our next question: what additional structures / conditions do we need to impose in order to be able to talk about exact sequences?

It turns out that no additional structures are needed but we need to impose additional conditions.

Exact sequences are about kernels, images and their coincidence. One can define them easily when we talk about modules but in the generality of additive categories, objects are not sets & morphisms are not maps, so we need to explain what we mean by kernels.

As usual, a recipe to define the kernels (and cokernels = quotients by images) are to look at their universal properties in the usual setting of abelian groups.

Let N, M be abelian groups & $\varphi: M \to N$ be a homomorphism. Let K be the kernel of φ and $\iota: K \hookrightarrow M$ be the inclusion. Then we have the following:

(*) $\forall\ L \in Ob(\mathbb{Z}\text{-Mod})$ & $\psi: L \to M$ a homom'm s.t. $\varphi \circ \psi = 0$ $\exists!\ \psi': L \to K$ making the following diagram commutative

$$\begin{tikzcd} L \arrow[d, dashed, "\psi'"] \arrow[dr, "\psi"] & \\ K \arrow[r, "\iota"] & M \end{tikzcd}$$

Definition (of kernel in an additive category) Let $\mathcal{C}$ be an additive category, $M, N \in Ob(\mathcal{C})$, $\varphi \in Hom_{\mathcal{C}}(M, N)$. By the kernel of φ we mean a pair (K, ι) w $K \in Ob(\mathcal{C})$, $\iota \in Hom_{\mathcal{C}}(K, M)$ s.t.

- $\varphi \circ \iota = 0$
- (K, ι) has a universal property that is a direct generalization of (*).

Definition (of cokernel in an additive category) The cokernel in $\mathcal{C}$ = the kernel in $\mathcal{C}^{opp}$. I.e. in the notation of the previous definition, we get a pair (C, π) w. $C \in Ob(\mathcal{C})$, $\pi \in Hom_{\mathcal{C}}(N, C)$ s.t.

- $\pi \circ \varphi = 0$

• and the universal property: $\forall\, \psi \in Hom_{\mathcal{C}}(N,L)$ s.t. $\psi \circ \varphi = 0$ $\exists!\, \psi' \in Hom_{\mathcal{C}}(C,L)$ s.t.

$$\begin{tikzcd} N \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ C \arrow[r, dashed, "\psi'"] & L \end{tikzcd}$$

is commutative.

Exercise: In the category of abelian groups, the cokernel of $\varphi: M \to N$ is $N/\operatorname{im}\varphi$ w. the projection $\pi: N \twoheadrightarrow N/\operatorname{im}\varphi$.

Definition: We say that $\varphi \in Hom_{\mathcal{C}}(M,N)$ is a **monomorphism** if $(0,0)$ is its kernel and is an **epimorphism** is $(0,0)$ is its cokernel.

For example, in A-Mod, monomorphism = injective & epimorphism = = surjective. Note that a monomorphism in $\mathcal{C}$ = epimorphism in $\mathcal{C}^{opp}$!

Exercise: • The following 2 conditions are equivalent

(a) $\varphi: M \to N$ is a monomorphism

(b) $\varphi \circ ?: Hom_A(L,M) \to Hom_A(L,N)$ is inj've $\forall\, L \in Ob(\mathcal{C})$

• Similarly, φ is an epimorphism $\iff$ $? \circ \varphi : Hom_A(N,L) \hookrightarrow Hom_A(M,L)$ $\forall\, L \in Ob(\mathcal{C})$.

• In particular, for any kernel (K, ι) we have that ι is a monomorphism & for any cokernel (C, π), π is an epimorphism.

Definition: We say that an additive category $\mathcal{C}$ is **abelian** if the following conditions hold:

(K) every morphism in $\mathcal{C}$ has a kernel

(C) every morphism in $\mathcal{C}$ has a cokernel

(M) for every monomorphism $\iota \in \mathrm{Hom}_{\mathcal{C}}(K, M)$ $\exists$ N & $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ s.t. (K, ι) is the kernel of φ.

(E) for every epimorphism $\pi \in \mathrm{Hom}_{\mathcal{C}}(N, C)$ $\exists$ M & $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ s.t. (C, π) is the cokernel of φ.

Example: A-Mod & A-Mod^{opp} are abelian categories.

Non-example: The category of free A-modules is not abelian if A is not a field. This is because every (not necessarily free) A-module is the cokernel (in the usual sense) of a linear map between free modules.

Example: A full subcategory of A-Mod (where A is an associative ring) that is closed under taking sub- & quotient modules is abelian. In particular, for A Noetherian, the category of fin. generated A-modules is abelian.

In an abelian category it makes sense to speak about subobjects of M (a pair of $K \in \mathrm{Ob}(\mathcal{C})$ & a monomorphism $\iota \in \mathrm{Hom}_{\mathcal{C}}(K, M)$) quotient objects etc. Axioms (M) & (E) ensure that these objects behave in a way we expect them to. In particular, it does make sense to talk about exact sequences.

Premium exer: in abelian category, isomorphism $\Leftrightarrow$ monomorphism & epimorphism