Lecture 24: Exactness & projective modules, II.

1) Projective modules
2) Flat modules.

Refs: [E], A.3.2, 6.1, 6.3.
BONUS: Injective modules.

1) Projective modules
1.1) Definition & equivalent characterizations.

Let P be an A-module. We know that the functor $\underline{Hom_A}(P,\cdot): A\text{-Mod} \to A\text{-Mod}$ is left exact, Sec 2.3 of Lec 23. A natural question is when (= for which P) it's exact, equivalently, (modulo the left exactness, Sec 2.4 of Lec 23), when:
$Hom_A(P,M) \twoheadrightarrow Hom_A(P,N)$ $\forall$ $M \twoheadrightarrow N$ (surjective A-linear map).

Definition: We say that P is a projective A-module if $\exists$ A-module P' and a set I s.t. $P\oplus P' \simeq A^{\oplus I}$

Example: Every free module is projective (take $P'=\{0\}$)

Later on we'll see that projective doesn't imply free. But projective modules are quite close to being free.
Here are equivalent characterization of projective modules.

Thm: TFAE

(1) P is projective.

(2) $\forall$ A-linear surjection $\pi: M \twoheadrightarrow P$ $\exists$ A-linear $\iota: P \to M$ s.t. $\pi \circ \iota = id_P$ (say that π splits).

(3) $\underline{Hom}_A(P, \cdot)$ is exact.

1.2) Proof of Theorem

(3) $\Rightarrow$ (2): $Hom_A(P,M) \xrightarrow{\pi\circ ?} Hom_A(P,P)$ is surjective $\Rightarrow$ $\exists$ $\iota \in Hom_A(P,M)$ s.t. $\pi \circ \iota = id_P$, which is (2).

(2) $\Rightarrow$ (1). Pick $\iota: M \to P$ w. $\pi \circ \iota = id_P$

In the proof we'll need the following.

Claim 1: $\pi\iota = id_P \Rightarrow M = \ker \pi \oplus \operatorname{im} \iota$

Proof of Claim:

$m = (m - \iota\pi(m)) + \iota(\pi(m))$ & $\pi(m - \iota\pi(m)) = \pi(m) - \pi\iota\pi(m) = [\pi\iota = id] = \pi(m) - \pi(m) = 0$ So $m - \iota\pi(m) \in \ker \pi$.

Hence $M = \ker\pi + \operatorname{im}\iota$. Now if $m \in \ker\pi \cap \operatorname{im}\iota$, then $m = \iota(n)$ & $n = \pi\iota(n) = \pi(m) = 0 \Rightarrow m = 0$. So $M = \ker\pi \oplus \operatorname{im}\iota$. $\square$

Now we get back to (2) $\Rightarrow$ (1). Pick generators p_i $(i \in I)$ of P giving $\pi: M := A^{\oplus I} \twoheadrightarrow P$, $(a_i)_{i\in I} \mapsto \sum_{i \in I} a_i p_i$. Note that $\pi\iota = id$ $\Rightarrow$ ι is injective $\Rightarrow$ $P \xrightarrow{\sim} \operatorname{im}\iota$. So can take $P' := \ker\pi$ arriving at $P \oplus P' \simeq A^{\oplus I}$.

$(1) \Rightarrow (3)$: We start with an auxiliary result:

Claim 2: Let $P_i, i \in I$, be some A-modules. Then

$(*)$ $\quad \underline{Hom}_A(\bigoplus_{i\in I} P_i, \cdot)$ is exact $\Leftrightarrow$ $\underline{Hom}_A(P_i, \cdot)$ is exact $\forall i \in I$.

Proof of Claim:

Recall, Rem. in Sec 1.2 of Lec 4, that we have a natural isomorphism $\eta_M: Hom_A(\bigoplus_{i\in I} P_i, M) \xrightarrow{\sim} \prod_{i\in I} Hom_A(P_i, M)$, $\tau \mapsto (\tau|_{P_i})_{i\in I}$.

These isomorphisms form a functor (iso)morphism, in particular, for $\varphi: M \twoheadrightarrow N$, the following diagram is commutative:

$$\begin{tikzcd} Hom_A(\bigoplus_{i\in I} P_i, M) \arrow[r, "\sim"] \arrow[d, "\varphi\circ ?"] & \prod_{i\in I} Hom_A(P_i, M) \arrow[d, "(\varphi\circ ?)_{i\in I}"] \\ Hom_A(\bigoplus_{i\in I} P_i, N) \arrow[r, "\sim"] & \prod_{i\in I} Hom_A(P_i, N) \end{tikzcd}$$

It follows that the left arrow is surjective iff the right is surjective iff $\forall\, i$ the map $Hom_A(P_i, M) \to Hom_A(P_i, N)$ is surjective.

Claim 2 follows. $\square$

Now we get back to proving $(1) \Rightarrow (3)$. Recall $\underline{Hom}_A(A, \cdot) \xrightarrow{\sim} id_{A\text{-Mod}}$, in particular, exact. By using "$\Leftarrow$" of $(*)$ for $P_i = A \; \forall\, i \in I$, we see that $\underline{Hom}_A(A^{\oplus I}, \cdot)$ is exact. Now by using "$\Rightarrow$" of $(*)$ for the decomposition $A^{\oplus I} = P \oplus P'$ we see that $\underline{Hom}_A(P, \cdot)$ is exact. $\square$

1.3) Projective vs free

How far are projective modules from being free? This depends on the ring A. Below we will only consider finitely generated projective modules.

We describe three results to be proved in Lecs 25 & 26.

Thm 1: Suppose A is local (i.e. has the unique maximal ideal). Then every finitely generated projective A-module is free.

There are other rings with this property. For example, the Quillen-Suslin theorem from 1976 states that this is also true for $A = \mathbb{F}[x_1, \dots x_n]$, where $\mathbb{F}$ is a field.

In general, (reasonable) projective modules can be characterized by a weaker property: they are "locally free." In order to define this condition we recall a special case of localization from Lec 9: $\forall$ prime ideal $\mathfrak{p} \subset A$, $S := A \setminus \mathfrak{p}$ is a multiplicative subset. Moreover, the ring $A_{\mathfrak{p}} := A[(A\setminus\mathfrak{p})^{-1}]$ is local (Prob 6 in HW2).

Definition: An A-module M is called locally free if $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module $\forall$ maximal ideal $\mathfrak{m} \subset A$.

Thm 2 (Serre): Let P be a finitely presented A-module. TFAE

1) P is projective
2) P is locally free.

Recall that being finitely presented means that $P \simeq A^{\oplus k}/\operatorname{im}\varphi$, where $\varphi: A^{\oplus \ell} \to A^{\oplus k}$ is A-linear (for some $k, \ell \in \mathbb{Z}_{\geq 0}$). Note that for Noetherian A, being finitely generated is equivalent to being finitely presented. Also for projective modules, fin. generated $\Leftrightarrow$ fin. presented.

Theorem 2 allows to give an elementary characterization of finitely generated projective modules over Dedekind domains – which leads to explicit examples of non-free projective modules.

Definition: Let A be a domain & M be an A-module. We say that M is torsion-free if $a \in A \setminus \{0\}$, $m \in M \setminus \{0\} \Rightarrow am \neq 0$

Example 1: • Every ideal $I \subset A$ is torsion-free but A/I is only torsion-free if $I = \{0\}$ or A.

• Every submodule of a free module is torsion-free. So $\forall$ projective module is torsion-free.

Thm 3: Let A be a Dedekind domain. Every torsion-free finitely generated module is projective.

Example 2: $A = \mathbb{Z}[\sqrt{-5}]$, $M = (2, 1+\sqrt{-5})$. Since M is an ideal in A, it's torsion-free, hence projective. On the other hand, it's not free. Indeed, this ideal is not principal, so $I \not\simeq A$. And since every two elements of I are linearly dependent ($a, b \in I \Rightarrow ba = ab$), we cannot have $I \not\simeq A^{\oplus k}$ for $k > 1$.

Remark: For more general domains, torsion-free doesn't imply projective, see Prob 5 in HW 5.

2) Flat modules.

Definition: An A-module F is **flat** if $F\otimes_A \bullet: A\text{-Mod} \to A\text{-Mod}$ is exact ($\Leftrightarrow$ sends injections to injections since, in general, $F\otimes_A \cdot$ is right exact).

Examples:

(I) $A^{\oplus I}$ is flat b/c $A^{\oplus I}\otimes_A \bullet \xrightarrow{\sim} \bullet^{\oplus I}$, this follows from our construction of tensor products in Sec 2.1 of Lec 20. If $N \hookrightarrow M$, then $N^{\oplus I} \hookrightarrow M^{\oplus I}$. So $A^{\oplus I}$ is flat.

(II) Recall, Sec 1.3 of Lec 21, that $(M_1\oplus M_2)\otimes_A N$ is naturally isomorphic to $M_1\otimes_A N \oplus M_2 \otimes_A N$. Then we can argue as in the proof of (1)$\Rightarrow$(3) in Sec 1.2 to show that:

$M_1\oplus M_2$ is flat $\Leftrightarrow$ both M_1, M_2 are flat.

Combining this with (I), we conclude that projective $\Rightarrow$ flat.

(III) Let $S\subset A$ be a multiplicative subset. Since $A[S^{-1}]\otimes_A \bullet \xrightarrow{\sim} \bullet[S^{-1}]$ (Sec 1.2 of Lec 22) & $\bullet[S^{-1}]$ is exact (Sec 2.3 of Lec 23), $A[S^{-1}]\otimes_A \bullet$ is exact, so $A[S^{-1}]$ is a flat A-module.

Remarks: 1) (III) gives an example of a flat module that is not projective: take $A=\mathbb{Z}$, $F:=\mathbb{Q}$, flat by III). The abelian group $\mathbb{Q}$ doesn't even admit a nonzero homom. to $\mathbb{Z}^{\oplus I}$ for any I. This is because $\forall\, \alpha\in\mathbb{Q}$, $n\in\mathbb{Z}_{>0}$ $\exists\, \alpha'\in\mathbb{Q}$ w. $\alpha=n\alpha'$ (such abelian groups are called divisible). On the other hand, if $(a_i)\in\mathbb{Z}^{\oplus I}$, $(a_i)\neq 0$, then we only have $\frac{1}{n}(a_i)\in\mathbb{Z}^{\oplus I}$ for finitely many positive integers.

2) On the other hand, the following fact holds:

Any finitely presented flat module is projective. A proof can be found in [E], Corollary 6.6.

BONUS: injective modules.

Let A be a commutative ring.

Definition: An A-module I is injective if $\mathrm{Hom}_A(\cdot, I)$: $A\text{-Mod}^{opp} \longrightarrow A\text{-Mod}$ is exact (equivalently, for an inclusion $N \hookrightarrow M$ the induced homomorphism

$$\mathrm{Hom}_A(I,M) \longrightarrow \mathrm{Hom}_A(I,N) \text{ is surjective}).$$

The definition looks very similar to that of projective modules, however the properties of injective & projective modules are very different! Projective modules - especially finitely generated ones - are nice, but injective modules are quite ugly, they are almost never finitely generated.

The simplest ring is $\mathbb{Z}$. Let's see what being injective means for $\mathbb{Z}$.

Definition: An abelian group M is divisible if $\forall\, m \in M, a \in \mathbb{Z}$ $\exists\, m' \in M$ s.t $am' = m$.

Example: The abelian group $\mathbb{Q}$ is divisible. So is $\mathbb{Q}/\mathbb{Z}$.

Proposition 1: For an abelian group M TFAE:

(a) M is injective

(b) M is divisible

Sketch of proof: (a) $\Rightarrow$ (b): apply

$$N \hookrightarrow M \Rightarrow \mathrm{Hom}_A(I, M) \twoheadrightarrow \mathrm{Hom}_A(I, N) \qquad (*)$$

to $M = \mathbb{Z}$, $N = a\mathbb{Z}$.

(b) $\Rightarrow$ (a) is more subtle. The first step is to show that if $(*)$ holds for $N \subset M$, then it holds for $N + \mathbb{Z}m \subset M$ $\forall\, m \in M$. So $(*)$ holds for all fin. gen'd submodules $N \subset M$. Then a clever use of transfinite induction yields $(*)$ for all submodules of M. $\square$

We can get examples of injective modules for more general rings as follows. Note that for an abelian group M, the group $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ is an A-module. The proof of the following is based on $\underline{\mathrm{Hom}_{\mathbb{Z}}}(A, \cdot)$ being right adjoint to the forgetful functor A-Mod $\to \mathbb{Z}$-Mod (Prob 2 in HW5). With this, the proof of the following is a premium exercise.

Proposition 2: If M is injective as an abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A,M)$ is an injective A-module.

Finally, using this proposition one can show that every A-module embeds into an injective one (the corresponding statement for projectives – that every module admits a surjection from a projective module – is easy b/c every free module is projective). This claim is important for Homological algebra.