Lecture 25: Exactness, projective & flat modules, III.

1) Nakayama Lemma.
2) Projective modules over local rings.
3) Projective vs locally free.

Ref: [AM], Sec 2.5.

BONUS: Why to care about Serre's Thm

0) Recap

In Lec 24 we discussed a connection between projective & free modules and stated three theorems to that effect.

Let A be a commutative ring

Thm 1: If A is local (= has unique max. ideal), then every finitely generated projective module is free.

Thm 2 (Serre): Let P be a finitely presented A-module. TFAE
1) P is projective
2) P is locally free, i.e. $P_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ $\forall$ max. ideal $\mathfrak{m} \subset A$.

In this lecture we'll prove Thm 1 & start proving Thm 2. We note that Thm 1 is a special case of 1) $\Rightarrow$ 2) in Thm 2. Indeed, if A is local w. max. ideal $\mathfrak{m}$, then all elements in $A \setminus \mathfrak{m}$ are invertible: if $a \notin \mathfrak{m} \Rightarrow (a) \not\subseteq \mathfrak{m} \Rightarrow (a) = A \Rightarrow a$ is invertible. Hence

$A_{\mathfrak{m}} = A$ & $P_{\mathfrak{m}} = P$ (cf. Sec 2.2 in Lec 9). But we'll use Thm 1 to prove Thm 2.

1) Nakayama lemma.

The proof of Thm 1 is based on a result of fundamental importance for Commutative algebra: Nakayama lemma. This is a key tool to study modules over local rings.

Thm (Nakayama lemma). Let A be a local ring with max. ideal $\mathfrak{m}$, M be a finitely generated A-module. If $\mathfrak{m}M = M$, then $M = \{0\}$.

Proof: Recall a Caley-Hamilton type lemma, Sec 1.1, Lec 11: if $I \subset A$ is an ideal & $\varphi: M \to M$ is an A-linear map s.t. $\operatorname{im}\varphi \subset IM$, then $\exists\, f(x) \in A[x]$ of the form $f(x) = x^n + a_1 x^{n-1} + \ldots + a_n$ s.t. $f(\varphi) = 0$ & $a_i \in I^i$.

Apply this to $I = \mathfrak{m}$ & $\varphi = id_M$ (so that $\operatorname{im}\varphi \subset IM$ thx to $M = \mathfrak{m}M$). We get $f(\varphi) = 0 \Rightarrow (1 + a_1 + a_2 + \ldots + a_n)\, id_M = 0$. But $a_i \in \mathfrak{m}$ $\forall i$ so $1 + (a_1 + \ldots + a_n) \in A$ is invertible, see the discussion after Thm 2. It follows that $id_M = 0 \Rightarrow M = \{0\}$. $\square$

Remark: One needs A to be local for the theorem to be true: take a field $\mathbb{F}$ & $A := \mathbb{F} \times \mathbb{F}$, $\mathfrak{m}, M = \{(x, 0) | x \in \mathbb{F}\}$. Then $\mathfrak{m}M = M$.

We'll need the following corollary of the theorem. We continue to assume that A is local w. maximal ideal $\mathfrak{m}$.

Corollary: Let M be a finitely generated A-module & $m_1, \ldots m_k \in M$. Let $\bar{m}_1, \ldots, \bar{m}_k$ be the images of $m_1, \ldots, m_k$ in $M/\mathfrak{m}M$. If $\bar{m}_1, \ldots, \bar{m}_k$ span the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$, then $m_1, \ldots, m_k$ span A-module M.

Proof: Set $N := \mathrm{Span}_A(m_1, \ldots m_k)$. Note that the composed map $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is surjective $\Leftrightarrow$ $M = N + \mathfrak{m}M \Leftrightarrow \forall\, m \in M\ \exists$ $a_1, \ldots a_k \in \mathfrak{m}, m_1, \ldots m_k \in M \mid m - \sum_{i=1}^{k} a_i m_i \in N \Leftrightarrow \mathfrak{m}(M/N) = M/N$. The A-module M/N is finitely generated. Applying the Nakayama lemma, we get $M/N = \{0\} \Leftrightarrow M = N$. $\square$

Exercise: Let M_1, M_2 be finitely generated A-modules & $\varphi \in \mathrm{Hom}_A(M_1, M_2)$. Then φ is surjective iff the induced map $M_1/\mathfrak{m}M_1 \to M_2/\mathfrak{m}M_2$ is surjective.

2) Projective modules over local rings.

Proof of Thm 1:

Proof: Let $\mathfrak{m} \subset A$ denote the maximal ideal, so $P/\mathfrak{m}P$ is a vector space over the field $A/\mathfrak{m}$. Since P is fin. generated over A, the vector space $P/\mathfrak{m}P$ is fin. dimensional. Let $\bar{m}_1, \ldots \bar{m}_\ell$ be a basis, and let $m_1, \ldots m_\ell$ be preimages of these elements in P (under $P \twoheadrightarrow P/\mathfrak{m}P$). By Corollary in Section 1, $P = \mathrm{Span}_A(m_1, \ldots, m_\ell)$, equivalently the homomorphism

$\pi: A^{\oplus \ell} \longrightarrow P, \ (a_1, \dots, a_\ell) \mapsto \sum_{i=1}^{\ell} a_i m_i$

is surjective. We want to show it's an isomorphism.

Note that $A^{\oplus \ell}/\mathfrak{m} A^{\oplus \ell}$ is naturally (in particular, $A/\mathfrak{m}$-linearly) identified w. $(A/\mathfrak{m})^{\oplus \ell}$. The homomorphism $(A/\mathfrak{m})^{\oplus \ell} \longrightarrow P/\mathfrak{m}P$ induced by π sends the standard basis e_i to the basis element $\overline{m}_i$, so is an isomorphism.

Since P is projective, π splits & $A^{\oplus \ell} \simeq P \oplus P'$ w. $P' := \ker \pi$ (Thm in Sec 1.1 & its proof in Sec 1.2 of Lec 24). It follows that

$$(A/\mathfrak{m})^{\oplus \ell} \simeq A^{\oplus \ell}/\mathfrak{m} A^{\oplus \ell} \simeq (P \oplus P')/\mathfrak{m}(P \oplus P') \simeq P/\mathfrak{m}P \oplus P'/\mathfrak{m}P'.$$

But $(A/\mathfrak{m})^{\oplus \ell}$ & $P/\mathfrak{m}P$ are isomorphic dim ℓ vector spaces over $A/\mathfrak{m}$. So $P'/\mathfrak{m}P' = \{0\}$. The A-module P' admits a surjective homomorphism from $A^{\oplus \ell}$. So, it's finitely generated.

Applying Nakayama lemma to $P' = \mathfrak{m}P'$ we see that $P' = \{0\}$. So π is an isomorphism. $\square$

3) Projective vs locally free.

In this section A is a general commutative ring.

3.1) Proof of 1) $\Rightarrow$ 2) in Thm 2

Since P is finitely generated, $\exists\ A^{\oplus n} \overset{\pi}{\twoheadrightarrow} P$. In the proof of 2) $\Rightarrow$ 1) in Sec 1.2 of Lec 24, we've seen that $A^{\oplus n} \xrightarrow{\sim} P \oplus P'$ w. $P' = \ker \pi$. Since localization commutes w. direct sums, Sec 1.2 of Lec 10, we have $A_{\mathfrak{m}}^{\oplus n} = P_{\mathfrak{m}} \oplus P'_{\mathfrak{m}}$. So $P_{\mathfrak{m}}$ is a fin. generated projective $A_{\mathfrak{m}}$-module. So by Thm 1, it's free.

3.2) Towards 2) $\Rightarrow$ 1)

Suppose P is a fin. presented A-module s.t. $P_{\mathfrak{m}}$ is a free $\forall\, \mathfrak{m}$. We want to show P is projective $\Leftrightarrow$ $\forall$ A-linear $M \underset{\varphi}{\twoheadrightarrow} N$, the induced map $\varphi \circ ?: \mathrm{Hom}_A(P,M) \to \mathrm{Hom}_A(P,N)$ is surjective.

Now let $\mathfrak{m} \subset A$ be a max. ideal so that $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module. Since the localization functor $\bullet_{\mathfrak{m}}$ is (right) exact, $\varphi_{\mathfrak{m}}: M_{\mathfrak{m}} \twoheadrightarrow N_{\mathfrak{m}}$, and since $P_{\mathfrak{m}}$ is free,

(1) $\qquad \varphi_{\mathfrak{m}} \circ ?: \mathrm{Hom}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}, M_{\mathfrak{m}}) \twoheadrightarrow \mathrm{Hom}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}, N_{\mathfrak{m}})$

The proof that the map $\varphi \circ ?$ is surjective breaks into two steps.

Step 1: From (1) we deduce that the linear map
$$(\varphi \circ ?)_{\mathfrak{m}}: \mathrm{Hom}_A(P,M)_{\mathfrak{m}} \longrightarrow \mathrm{Hom}_A(P,N)_{\mathfrak{m}}$$
(obtained from $\varphi \circ ?$ by applying the functor $\bullet_{\mathfrak{m}}$) is surjective.

Step 2: We will prove the following: let $\tilde{\varphi}: \tilde{M} \to \tilde{N}$ be an A-linear map. If $\tilde{\varphi}_{\mathfrak{m}}: \tilde{M}_{\mathfrak{m}} \to \tilde{N}_{\mathfrak{m}}$ is surjective for all max. ideals $\mathfrak{m} \subset A$, then $\tilde{\varphi}$ is surjective.

Then we apply Step 2 to $\tilde{M} = \mathrm{Hom}_A(P,M)$, $\tilde{N} = \mathrm{Hom}_A(P,N)$ & $\tilde{\varphi} = \varphi \circ ?$ and use Step 1 to complete the proof that $\varphi \circ ?$ is surjective.

Proof of Step 1: Recall (Prob 7 in HW2) that for any finitely presented A-module P, any A-module L & any multiplicative subset $S \subset A$ we have the following commutative diagram (which is exactly where the condition of being finitely presented is used)

$$\begin{tikzcd}
& \mathrm{Hom}_A(P,L) \arrow[dl, "\psi \mapsto \frac{\psi}{1}"'] \arrow[dr, "\psi \mapsto \psi[S^{-1}]"] & \\
\mathrm{Hom}_A(P,L)[S^{-1}] \arrow[rr, dashed, "\sim"] & & \mathrm{Hom}_{A[S^{-1}]}(P[S^{-1}], L[S^{-1}])
\end{tikzcd}$$

Here the horizontal map comes from the universal property of localization (Sec 2.2, Lec 9) so is uniquely determined by the commutativity of the diagram. Hence it must be given by

(2) $\qquad \frac{\psi}{s} \mapsto \frac{1}{s}\psi[S^{-1}]$

Now we claim that for any linear map $\varphi: M \to N$ the following diagram is commutative (where we specialize to $S:=A\setminus\mathfrak{m}$).

$$\begin{tikzcd}
\mathrm{Hom}_A(P,M)_{\mathfrak{m}} \arrow[r, "\sim", "\frac{\psi}{s}\mapsto\frac{1}{s}\psi_{\mathfrak{m}}"'] \arrow[d, "(\varphi\circ ?)_{\mathfrak{m}}"] & \mathrm{Hom}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}, M_{\mathfrak{m}}) \arrow[d, "\varphi_{\mathfrak{m}}\circ ?"] \\
\mathrm{Hom}_A(P,N)_{\mathfrak{m}} \arrow[r, "\sim", "\frac{\psi'}{s}\mapsto\frac{1}{s}\psi'_{\mathfrak{m}}"'] & \mathrm{Hom}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}, N_{\mathfrak{m}})
\end{tikzcd}$$

$\downarrow\longrightarrow$: recall that for $\tilde{\varphi}: \tilde{M} \to \tilde{N}$, the map $\tilde{\varphi}_{\mathfrak{m}}$ is given by: $\frac{\tilde{m}}{s} \mapsto \frac{\tilde{\varphi}(\tilde{m})}{s}$

So $(\varphi\circ ?)_{\mathfrak{m}}$ sends $\frac{\psi}{s}$ to $\frac{\varphi\circ\psi}{s}$. And then the horizontal map sends this to $\frac{1}{s}(\varphi\circ\psi)_{\mathfrak{m}}$

$\longrightarrow\downarrow$: $\frac{\psi}{s} \mapsto \varphi_{\mathfrak{m}}\circ(\frac{1}{s}\psi_{\mathfrak{m}})$ = [composition is bilinear] $= \frac{1}{s}\varphi_{\mathfrak{m}}\circ\psi_{\mathfrak{m}}$ [$\cdot_{\mathfrak{m}}$ is a functor] $= \frac{1}{s}(\varphi\circ\psi)_{\mathfrak{m}}$.

Commutativity of the diagram & (1) imply Step 1

BONUS 1: Why to care about Serre's Thm, briefly.

Being projective is an important "niceness" property from the point of category theory (Hom from such object behaves well). It turns out that being locally free is important geometrically - it says that this module corresponds to a "vector bundle". Below is a short account on this.

Let $X \subset \mathbb{F}^n$ be an algebraic subset (Sec. 1.4 of Lec 14) & $A = \mathbb{F}[X]$ be its algebra of polynomial functions (Sec 2 of Lec 15). The maximal ideals in A are in bijection with the points of X & if $\mathfrak{m} \subset A$ corresponds to $x \in X$, then $A_{\mathfrak{m}}$ "controls" the behavior of X locally near x (see Sec. 3 of Lec 15)

A vector bundle on a "space" (which could mean a topological space, a C^∞-manifold, or an algebraic variety) is roughly speaking an assignment that sends a point to a vector space & we want this space to depend on a point "locally trivially". For example, in the study of C^∞-manifolds we look at the tangent bundle (to each point of a manifold we assign its tangent space) or the exterior powers of cotangent bundles. In all contexts vector bundles are among the most important structures on manifolds.

In the context of polynomial subsets of $\mathbb{F}^n$ vector bundles, by definition, come from <u>locally free</u> (finitely generated) A-modules (a module M attaches the space $M/\mathfrak{m}M$ to $x \in X$ corresponding to $\mathfrak{m}$). The fact that these modules are exactly the projective ones is very important for the interplay between Algebraic geometry &

Homological algebra.