Lecture 26: exactness, projective & flat modules, IV

1) Completion of proof of Serre's theorem
2) Projective modules over Dedekind domains
3) Invertible modules

1) Completion of proof of Serre's theorem

Let A be a commutative ring.
In Sec 1.3 of Lec 24 we have stated:

Thm 2 (Serre): Let P be a finitely presented A-module. TFAE
1) P is projective
2) P is locally free, i.e. $P_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ $\forall$ max. ideal $\mathfrak{m}\subset A$.

In Sec 3 of Lec 25 we've proved 1)$\Rightarrow$2) and reduced 2)$\Rightarrow$1) to the following (Step 2 in Sec 3.2)

Lemma: Let $\tilde{M},\tilde{N}$ be A-modules & $\tilde{\varphi}:\tilde{M}\to\tilde{N}$ be A-linear map. If $\varphi_{\mathfrak{m}}:\tilde{M}_{\mathfrak{m}}\to\tilde{N}_{\mathfrak{m}}$ is surjective $\forall$ max. ideals $\mathfrak{m}\subset A$, then $\tilde{\varphi}$ is surjective.

Proof: Step 1: assume C is an A-module s.t. $C_{\mathfrak{m}} = \{0\}$ $\forall$ max. ideals $\mathfrak{m} \subset A$. We claim that $C = \{0\}$. Indeed, take $c \in C$. Since $C_{\mathfrak{m}} = \{0\}$, we have $\frac{c}{1} = \frac{0}{1} \Leftrightarrow \exists\, u \in A \setminus \mathfrak{m} \mid uc = 0$ in C. Consider the subset $Ann_A(c) = \{a \in A \mid ac = 0\}$. This is an ideal (exercise). If $c \neq 0$, then $1 \notin Ann_A(c) \Rightarrow \exists\, \mathfrak{m}$ s.t. $Ann_A(c) \subset \mathfrak{m}$. But we've seen $\exists\, u \in Ann_A(c) \setminus \mathfrak{m}$, contradiction.

Step 2: Set $C := \tilde{N}/\operatorname{im}\tilde{\varphi}$. By Sec 1.1 of Lec 10:

- $\operatorname{im}(\tilde{\varphi}_{\mathfrak{m}}) = (\operatorname{im}\tilde{\varphi})_{\mathfrak{m}}$ $\forall$ max. ideal $\mathfrak{m} \subset A$ &
- $(\tilde{N}/K)_{\mathfrak{m}} \simeq \tilde{N}_{\mathfrak{m}}/K_{\mathfrak{m}}$ $\forall$ submodule $K \subset \tilde{N}$ (e.g. $K = \operatorname{im}\tilde{\varphi}$) & $\forall\, \mathfrak{m}$

Since $\tilde{\varphi}_{\mathfrak{m}}$ is surjective, we see that $C_{\mathfrak{m}} = \{0\}$ $\forall$ max. ideal $\mathfrak{m}$. By Step 1, $C = \{0\}$ finishing the proof. $\square$

2) Projective modules over Dedekind domains

In this section A is a Dedekind domain. We'll need two facts about A:

1) A is Noetherian (by definition)
2) $\forall$ max. ideal $\mathfrak{m} \subset A$, the localization $A_{\mathfrak{m}}$ is a PID (Problem 3 in HW 3)

In Sec 1.3 of Lec 24 we've stated the following

Theorem: Let M be a finitely generated A-module. If M is torsion-free ($a\in A\setminus\{0\}, m\in M\setminus\{0\} \Rightarrow am\neq 0$), then M is projective.

Proof:

Step 1: We first consider a special case: A is PID. Here we have a complete classification of finitely generated modules, Sec 3.3 of Lec 6: $M\simeq A^{\oplus \ell}\oplus \bigoplus_{i=1}^{k} A/(p_i^{d_i})$. Such a module is torsion-free iff $k=0$ (b/c $p_i^{d_i}(1+(p_i^{d_i}))=0$). In particular, if M is torsion-free, then it's free.

Step 2: Since A is Noetherian, P is finitely presented. So we can use the Serre theorem: P is projective iff it's locally free. Note that $A_{\mathfrak{m}}$ is a PID so, thx to Step 1, it's sufficient to prove $P_{\mathfrak{m}}$ is torsion-free.

Step 3: We'll prove a more general claim: if A is a domain, M is a torsion-free A-module, then $M[S^{-1}]$ is a torsion-free $A[S^{-1}]$-module $\forall$ multiplicative subset $S\subset A\setminus\{0\}$. Indeed, take non-zero elements $\frac{a}{s}\in A[S^{-1}]$, $\frac{m}{t}\in M[S^{-1}]$. If $\frac{am}{st}=\frac{0}{1}$, then $\exists\, u\in S$ s.t. $uam=0$. But $u,a\neq 0 \Rightarrow ua\neq 0$ & $m\neq 0$ leads to a contradiction. $\square$

3) Invertible modules

3.1) Definition & examples

Let A be a commutative ring.

Definition: An A-module L is called *invertible* if $\exists$ A-module L' w. $L\otimes_A L'\simeq A$ (an A-module isomorphism).

Examples: 1) Let A be a field $\mathbb{F}$. We claim that an $\mathbb{F}$-vector space L is invertible $\Leftrightarrow \dim_{\mathbb{F}} L=1$. This follows from the computation of tensor products of free modules in Sec 2.3 of Lec 20: if $e_i, i\in I$, is a basis of L & $e_j', j\in J$, is a basis of L', then $e_i\otimes e_j'$ form a basis in $L\otimes_A L'$ so, in our case, $|I\times J|=1 \Rightarrow |I|=1$.

2) Let A be a Dedekind domain, and L be an ideal. We claim it is an invertible A-module.

By Thm in Sec 2, L is a projective, hence flat A-module. In particular, if L' is another ideal, then the inclusion $L'\hookrightarrow A$ gives rise to $L\otimes_A L'\hookrightarrow L\otimes_A A\xrightarrow{\sim} L$ and the map is given by $l\otimes l'\mapsto ll'$, so its image is LL', the product of ideals. We claim that we can find L' such that LL' is a principal ideal so that $L\otimes_A L'\xrightarrow{\sim} LL'\xrightarrow{\sim} A$. For this, we decompose L into the product of prime ideals $\mathfrak{p}_1 \dots \mathfrak{p}_k$. It's enough to find $\mathfrak{p}_i'$ s.t. $\mathfrak{p}_i \mathfrak{p}_i'$ is principal. Take $a \in \mathfrak{p}_i \setminus \{0\}$. Then $\mathfrak{p}_i$ occurs in the decomposition of (a) (exercise): $(a) = \mathfrak{p}_i \mathfrak{q}_1 \dots \mathfrak{q}_\ell$ & we take $\mathfrak{p}_i' = \mathfrak{q}_1 \dots \mathfrak{q}_\ell$. This finishes the proof of the claim that L is invertible. And, in fact, every invertible A-module is isomorphic to some ideal in A, as we will sketch below.

Rem: Here are motivations to consider invertible modules. A categorical motivation is that the functor $L \otimes_A \bullet : A\text{-Mod} \to A\text{-Mod}$ has an "inverse", $L' \otimes_A \bullet$ – by algebra properties of tensor products. So these functors can be viewed as "symmetries" of A-Mod.

A geometric motivation is that counterparts of invertible modules, line bundles, a special class of vector bundles, are extremely important, in Algebraic geometry.

Side remark: using invertible modules and the description of invertible modules over Dedekind domains, one can generalize the class group of a Dedekind domain to arbitrary rings. We consider the Picard group: its elements are isomorphism classes of invertible modules and the product comes from taking the tensor products

3.2) Properties of invertible modules

Proposition: Every invertible module is projective.

Proof:

Let L, L' be s.t. $L\otimes_A L' \xrightarrow{\sim} A$, let ι denote an isomorphism. We are going to establish a functor isomorphism $\underline{Hom}_A(L,\cdot) \xrightarrow{\sim} L'\otimes_A \cdot$. The functor $L'\otimes_A \cdot$ is right exact, so this would imply L is projective.

In the proof we will need the following construction. For A-modules L_1, M, N consider the A-linear map

$$\theta_{L_1}^{M,N}: Hom_A(M,N) \longrightarrow Hom_A(L_1\otimes_A M, L_1\otimes_A N),\ \varphi \mapsto id_{L_1}\otimes\varphi.$$

Claim: If L' is invertible, then $\theta_{L'}^{M,N}$ is an isomorphism.

With this claim the proof is as follows: we get isomorphisms $\theta_{L'}^{L,N}: Hom_A(L,N) \xrightarrow{\sim} Hom_A(L'\otimes_A L, L'\otimes_A N)$ that constitute a functor morphism (exercise) and then use that $L'\otimes_A L \xrightarrow{\sim} A$ & functor isomorphism $\underline{Hom}_A(A,\cdot) \xrightarrow{\sim} id_{A\text{-Mod}}$ to get $\underline{Hom}_A(L,\cdot) \xrightarrow{\sim} L'\otimes_A \cdot$.

Proof of Claim: Observe that:

(i) $\theta_A^{M,N}$ becomes the identity under the identifications $A\otimes_A M \xrightarrow{\sim} M$, $A\otimes_A N \xrightarrow{\sim} N$.

(ii) $\theta_{L_1\otimes_A L_2}^{M,N} = \theta_{L_1}^{L_2\otimes_A M, L_2\otimes_A N}\circ\theta_{L_2}^{M,N}$: both sides send $\varphi\in Hom_A(M,N)$

to $\mathrm{id}_{L_1\otimes_A L_2}\otimes\varphi = \mathrm{id}_{L_1}\otimes \mathrm{id}_{L_2}\otimes \varphi\in \mathrm{Hom}_A(L_1\otimes_A L_2\otimes_A M, L_1\otimes_A L_2\otimes_A N)$

Apply (ii) to $L_1=L, L_2=L'$. Since $L\otimes_A L'\xrightarrow{\sim} A$, by (i), $\theta^{M,N}_{L\otimes_A L'}$ is an isomorphism. So (ii) implies $\theta^{M,N}_{L'}$ is injective $\forall M,N$. But L, L' play symmetric roles, so for $M'=L'\otimes_A M$, $N'=L'\otimes_A N$, the map $\theta^{M',N'}_{L}$ is injective as well. If the composition of two injections is a bijection, then both injections are bijections. $\square$ of Claim.

$\square$ of Proposition

Corollary: Let A be a Dedekind domain & L be an invertible A-module. If L is finitely generated, then it's isomorphic to an ideal in A.

Sketch of proof: We'll prove that L embeds into $\mathrm{Frac}(A)$ as an A-submodule, so it's isomorphic to a fractional ideal (Sec 1.1 of Lec 13). Every fractional ideal is isomorphic to an actual one: an isomorphism is given by multiplying w. suitable element of A.

Since L is projective it's torsion-free so the natural homomorphism $L\longrightarrow L[S^{-1}]$ is an embedding for every multiplicative $S\subset A$ including $S:=A\setminus\{0\}$. So we need to show that for this S, we have $L[S^{-1}]\xrightarrow{\sim}\mathrm{Frac}(A)$. Thx to Example 1) in Sec 3.1, this will follow once we show that $L[S^{-1}]$ is invertible as an $A[S^{-1}]$-module. This in turn follows from:

Exercise: For any A-modules M, N, we have a natural isomorphism $M[S^{-1}] \otimes_{A[S^{-1}]} N[S^{-1}] \xrightarrow{\sim} (M \otimes_A N)[S^{-1}]$. Hint: use universal properties to produce homomorphisms in both directions.

$\square$

Remark: In fact, every invertible module over a Noetherian ring is automatically finitely generated, equivalently, Noetherian module. The shortest proof is categorical: the functor $L \otimes_A \bullet : A\text{-Mod} \longrightarrow A\text{-Mod}$ is an "equivalence" of "abelian categories" & being Noetherian is a property inherent to an abelian category so doesn't change under equivalence. Since L is the image of a Noetherian object, A, it's also Noetherian.