Lecture 6: Noetherian rings & modules, II / Modules over PID, I.

1) Further properties of Noetherian modules.
2) Artinian modules & rings.
3) PID's & their modules

References: [AM], Chapters 6-8 (for 1) & 2));
[V] Sec 9.3, Dummit & Foote, Ch. 12 for 3).

1) Further properties of Noetherian modules.

Let A be a ring (may not be Noetherian) & M be A-module. In Sec 3 of Lec 5 we've stated:

Proposition: Let $N \subset M$ be a submodule TFAE:

(1) M is Noetherian

(2) Both N, M/N are Noetherian.

Proof:

(1) $\Rightarrow$ (2): M is Noetherian $\Rightarrow$ N is Noeth'n (tautology)
Check M/N is Noetherian by verifying AC termination. Let $\pi: M \twoheadrightarrow M/N$, $m \mapsto m+N$.

Let $(\underline{N}_i)_{i\geqslant 0}$ be an AC of submodules in M/N, $N_i := \pi^{-1}(\underline{N}_i)$
$\underline{N}_i \subset \underline{N}_{i+1} \Rightarrow N_i \subset N_{i+1}$ so $(N_i)_{i\geqslant 0}$ form an AC of submodules of M, it must terminate: $\exists\, k \geqslant 0 \mid N_j = N_k \ \forall j \geqslant k$. But $\underline{N}_i = \pi(N_i)$ so $\underline{N}_j = \pi(N_j) = \pi(N_k) = \underline{N}_k$. So $(\underline{N}_i)_{i\geqslant 0}$ terminates.

(2) $\Rightarrow$ (1): Have $(N_i)_{i\geqslant 0}$ is an AC of submodules in M. Then $(N_i \cap N)_{i\geqslant 0}$ is AC in N & $(\pi(N_i))_{i\geqslant 0}$ is AC in M/N.

We know that both terminate $\Rightarrow$
$\exists\, k>0$ s.t. $N_j \cap N = N_k \cap N$ & $\pi(N_j) = \pi(N_k)$ $\forall j > k$.
Want to check: $N_j = N_k$ (so (N_i) terminates):
$n \in N_j \rightsquigarrow \pi(n) \in \pi(N_j) = \pi(N_k)$ so $\exists\, n' \in N_k \mid \pi(n') = \pi(n)$
$\Leftrightarrow \pi(n-n') = 0 \Leftrightarrow n - n' \in N$. But $n, n' \in N_j$ (b/c $n' \in N_k \subset N_j$) $\Rightarrow$
$n-n' \in N_j \Rightarrow n - n' \in N \cap N_j = N \cap N_k \Rightarrow n = n' + (n-n') \in N_k$ b/c
both summands are in N_k. This shows $N_j = N_k$. $\square$

2) Artinian modules & rings.

2.1) Definition of Artinian modules

Recall: Noetherian $\Leftrightarrow$ satisfies AC condition (Sec 1.2 of Lec 5)

Definition: Let M be A-module. A descending chain (DC) of submodules is $(N_i)_{i \geq 0}$ s.t. $N_k \supseteq N_{k+1}$ $\forall k \geq 0$.

Definition: M is an Artinian A-module if $\forall$ DC of submodules terminates ("DC termination")

Example: $A = \mathbb{F}$ (a field). Claim: Artinian $\Leftrightarrow$ finite dim'l.
$\Leftarrow$: is clear b/c dimensions decrease in DC's.
$\Rightarrow$: let $\dim M = \infty \Leftrightarrow \exists$ lin. indep. vectors $m_i \in M$, $i \geq 0$.
Define $N_j = \mathrm{Span}_{\mathbb{F}}(m_i \mid i \geq j)$ – a DC of subspaces that doesn't terminate.

2.2) Basic properties.

The first result is analogous to Proposition in Sec 1.2 of Lec 5, the proof is exercise.

Proposition 1: For A-module M TFAE:

1) M is Artinian
2) $\forall$ nonempty set of submodules of M has a min'l el-t (w.r.t. $\subset$)

Proposition 2: M is A-module, $N \subset M$ is an A-submodule.
TFAE: 1) M is Artinian.
2) Both N & M/N are Artinian.

Proofs: repeat those in Noeth'n case from Sec 1 (exercise).

2.3) Artinian rings.

Definition: A ring A is Artinian if it's Artinian as A-module.

Examples: 1) Any field $\mathbb{F}$ is Artinian. More generally, let A be an $\mathbb{F}$-algebra s.t. $\dim_{\mathbb{F}} A < \infty$. Then A is Artinian ring (b/c A-submodule is a subspace). E.g. we can take $A = \mathbb{F}[x]/(f)$ $\forall$ nonzero $f \in \mathbb{F}[x]$ or $A = \mathbb{F}[x,y]/(x^2, xy, y^2)$.

3) $A = \mathbb{Z}/n\mathbb{Z}$ is Artinian (b/c it's a finite set so every DC of subsets terminates).

4) Every nonzero el-t, a, of Artinian ring is either invertible or zero-divisor. Indeed, let $a \in A$, then $(a) \supseteq (a^2) \supseteq (a^3) \supseteq \dots$

a DC of ideals. It terminates
$(a^k)=(a^{k+1}) \Rightarrow \exists\, b\in A$ s.t. $a^k = ba^{k+1} \Leftrightarrow (1-ab)a^k=0$
$\Leftrightarrow$ a is zero divisor or $1=ab$ (so a is invertible).

Remark: In particular, every Artinian domain is a field. So 4) gives a lot of non-examples of Artinian rings (e.g. $\mathbb{Z}$). This is a stark contrast with the Noetherian condition, where "almost any reasonable" ring is Noetherian.

Here's a further interesting fact:

Thm: Every Artinian ring is Noetherian.

For proof, see [AM], Prop 8.1 - Thm 8.5 (comments: nilradical $=\sqrt{0}=$ $= \bigcap$ all prime ideals by Prop. 1.8, Jacobson radical $=\bigcap$ all max. ideals).

2.4) Finite length modules.

Thm motivates us to consider modules that are both Noetherian (AC termin'n) & Artinian (DC termin'n) so satisfy ("AC & DC" termin'n). They admit an equivalent characterization.

Definition: Let M be an A-module.

i) Say that M is simple if $\{0\}\neq M$ are the only two submodules of M.

ii) Let M be arbitrary. By a filtration (by submodules) on M we mean $\{0\}=M_0\subset M_1\subset M_2\subset\ldots\subset M_k=M$ (finite AC of submodules).

iii) A Jordan-Hölder (JH) filtr'n is a filtr'n $\{0\} = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_k = M$ s.t. M_i / M_{i-1} is simple $\forall i$ (so a JH filtr'n is "tightest possible")

iv) M has finite length if a JH filtr'n exists.

Example: 1) When $A = \mathbb{F}$ is a field, an A-module M is simple $\iff \dim_{\mathbb{F}} M = 1$.

2) Let $A = \mathbb{Z}$ & consider the A-module $M = \mathbb{Z}/4\mathbb{Z}$. It's JH filtration is $M_0 = \{0\}$, $M_1 = 2\mathbb{Z}/4\mathbb{Z}$, $M_2 = M$.

Proposition: For an A-module M TFAE:

1) M is Artinian & Noetherian.

2) M has finite length.

Proof: 2) $\Rightarrow$ 1): M has fin length $\rightsquigarrow$ JH filtr'n $\{0\} = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_k = M$. We prove by induction on i that M_i is Artinian & Noetherian.

Base: $i=1$: M_1 is simple $\Rightarrow$ Artinian & Noetherian.

Step: $i-1 \rightsquigarrow i$: M_{i-1} is Art'n & Noeth'n, so is M_i / M_{i-1} b/c it's simple. $\Rightarrow$ by Prop in Sec 1 M_i is Noetherian & by Prop 2 in Sec. 2.1, M_i is Artinian.

Use this for $i = k \rightsquigarrow M_i = M$ is Artinian & Noeth'n. So 2) $\Rightarrow$ 1).

1 $\Rightarrow$ 2): M is Artinian & Noetherian. Want to produce a JH filtr'n. By induction: $M_0 = \{0\}$.

Suppose we've constr'd $M_i \subset M$. Need M_{i+1}.
Note: M/M_i is Artinian & therefore $\forall$ nonempty set of submodules has a min el't. Assume $M_i \neq M$. Consider the set of all nonzero submodules of M/M_i. It's $\neq \emptyset$ so has a min'l element, N. This N must be simple. Now take M_{i+1} to be the preimage of N under $M \twoheadrightarrow M/M_i$.
So $M_{i+1}/M_i \simeq N$, simple.

We've got is an AC $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots$, it must terminate b/c M is Noeth'n. By constr'n it can only terminate at $M_i = M$. So we've got a JH filtration $\square$

Exercise: We can classify simple modules as follows: a map $\mathfrak{m} \mapsto A/\mathfrak{m}$ defines a bijection between the set of maximal ideals in A and the set of simple A-modules (up to isomorphism).

3) PID's & their modules

A highlight of the study of finitely generated abelian groups is their classification – we can completely describe all possible fin. gen'd abelian groups. Recall that an abelian group is the same thing as a $\mathbb{Z}$-module. One can ask: for which rings one can fully classify all finitely generated modules. Turns out that this rarely happens.
Here's a class of rings for which the classification is possible.

3.1) Definition & examples

Definition: A ring A is a principal ideal domain (PID) if it's a domain & $\forall$ ideal is principal (generated by one element)

Examples: • $\mathbb{Z}$, $\mathbb{F}[x]$ ($\mathbb{F}$ is field) are PID's; $\forall$ "Euclidian domain" ($\approx$ can divide w. remainder) is PID's (e.g. $\mathbb{Z}[i]$); $\mathbb{F}[[x]]$ is a PID by Problem 1 in HW1

Non-examples: $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[x]$, $\mathbb{F}[x,y]$ are not PID:

$$\mathbb{Z}[\sqrt{-5}] \supset (2, 1+\sqrt{-5}), \quad \mathbb{Z}[x] \supset (2,x), \quad \mathbb{F}[x,y] \supset (x,y)$$ - not principal.

3.2) Properties of PID's.

Let A be a PID. Take $a_1, \ldots a_n \in A \rightsquigarrow$ ideal $(a_1,\ldots a_n) \in A$
$\exists\, d \in A \,|\, (a_1,\ldots a_n) = (d)$, defined uniquely up to invertible factor $\Rightarrow$

- d divides $a_1,\ldots,a_n$ b/c $a_1,\ldots,a_n \in (d)$.
- if d' divides $a_1,\ldots,a_n \Rightarrow d'$ divides d $(=\sum_{i=1}^n x_i a_i$ for some $x_1,\ldots x_n \in A)$.

Besause of these, d is called the GCD of $a_1,\ldots a_n$.

Classical application of GCD: PID $\Rightarrow$ UFD.

Remark PID $\Rightarrow$ Noetherian.

3.3) Classification of modules

Let A be PID. Let M be a finitely generated A-module.

Thm: 1) $\exists\, k\in\mathbb{Z}_{\geqslant 0}$, primes $p_1,\dots,p_\ell\in A$; $d_1,\dots,d_\ell\in\mathbb{Z}_{>0}$ s.t

$$M\simeq A^{\oplus k}\oplus\bigoplus_{i=1}^{\ell} A/(p_i^{d_i}).$$

2) k is uniquely determined by M, $(p_1^{d_1}),\dots,(p_\ell^{d_\ell})$ are uniquely determined up to permutation.

Example: For $A=\mathbb{Z}$, Thm = classif'n of fin. gen'd abelian grps.

3.4) Case of $A=\mathbb{F}[x]$, $\mathbb{F}$ is alg. closed.

Assume $\dim_{\mathbb{F}} M<\infty$ (so $k=0$) & $\mathbb{F}$ is alg. closed $\Rightarrow$ primes in $\mathbb{F}[x]$ are $x-\lambda$, $\lambda\in\mathbb{F}$, (up to invertible factor).

Main Thm $\Rightarrow$ $\exists\ \lambda_i\in\mathbb{F}$, $d_i\in\mathbb{Z}_{>0}$ s.t. $M\simeq\bigoplus_{i=1}^{\ell}\mathbb{F}[x]/((x-\lambda_i)^{d_i})$

Reminder (Lec 3, Sec 2.2)

A module over $\mathbb{F}[x]$ = $\mathbb{F}$-vector space & an operator X. For a fixed $\mathbb{F}$-vector space M, operators $X_M, X_M': M\to M$ give isomorphic $\mathbb{F}[x]$-module structures $\Leftrightarrow$ X_M, X_M' are conjugate: $\varphi: M\to M$ is a homomorphism between the 2 module structures iff $\varphi\circ X_M=X_{M'}\circ\varphi$ so φ is an isomorphism $\Leftrightarrow$ $\varphi X_M\varphi^{-1}=X_M'$. So the Main Thm allows to classify linear operators up to conjugation.

Choose an $\mathbb{F}$-basis in $\mathbb{F}[x]/((x-\lambda_i)^{d_i})$: $(x-\lambda_i)^j$, $j=0,\dots,d_i-1$.

$$X(x-\lambda_i)^j = [x=(x-\lambda_i)+\lambda_i] = \begin{cases} (x-\lambda_i)^{j+1} + \lambda_i (x-\lambda_i)^j & \text{if } j < d_i - 1 \\ \lambda_i (x-\lambda_i)^j & \text{if } j = d_i - 1. \end{cases}$$

So X acts as a Jordan block:

$$J_{d_i}(\lambda_i) = \begin{pmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & 1 & \ddots & \\ 0 & & 1 & \lambda_i \end{pmatrix}$$

Main Thm in this case is:

Jordan Normal Form thm:

Let X be a linear operator on a fin. dim. $\mathbb{F}$-vector space, M, where $\mathbb{F}$ is alg. closed. Then in some basis X is represented by a "Jordan matrix": $\mathrm{diag}(J_{d_1}(\lambda_1), \dots, J_{d_e}(\lambda_e))$.

Can recover the pairs $(d_1, \lambda_1), \dots, (d_e, \lambda_e)$ from X - will discuss in Lec 7.