Lecture 8: localization of rings & modules, I

1) Completion of proof from last lecture
2) Localization of rings.
See refs for Lec 7; + [AM], Intro to Sec 3.

1) Completion of proof from last lecture

Let A be a PID & M be a finitely generated A-module. We already know that
$$M \simeq A^{\oplus k} \oplus \bigoplus_{i=1}^{k} A/(p_i^{d_i})$$
for some $k \geqslant 0$, primes $p_1, \dots p_\ell \in A$ & $d_1, \dots d_\ell \in \mathbb{Z}_{>0}$ & we want to recover these from M. For this purpose in Sec 1.5 of Lec 7 we defined, for a prime $p \in A$ & $s \in \mathbb{Z}_{\geqslant 0}$
$$d_{p,s}(M) = \dim_{A/(p)} p^s M / p^{s+1} M$$
and stated the following proposition whose proof will complete the proof of Thm in Sec 3.3 of Lec 6.

Proposition: For $M \simeq A^{\oplus k} \oplus \bigoplus_{i=1}^{\ell} A/(p_i^{d_i})$, we have
$$d_{p,s}(M) = k + \#\{i \mid (p_i) = (p) \ \&\ d_i > s\}.$$

Proof of Proposition:
Step 1: explain how $d_{p,s}$ behaves on direct sums:

Claim: $d_{p,s}(\bigoplus_{i=1}^{r} M_i) = \sum_{i=1}^{r} d_{p,s}(M_i)$ $\forall$ fin. generated A-modules M_i, $i=1, \dots r$.

Proof of the claim: Enough to consider $r=2$.

$p^s(M_1 \oplus M_2)/p^{s+1}(M_1 \oplus M_2) \simeq$ [similar to Problem 5 in HW1, exercise]

$p^s M_1/p^{s+1}M_1 \oplus p^s M_2/p^{s+1}M_2$

and the claim follows: the dimension of the direct sum of vector spaces is the sum of dimensions of summands

Step 2: Need to compute $d_{p,s}$ of possible summands of M:

$A,\ A/(p^t),\ A/(q^t),\ (q) \neq (p)$.

i) A:

$$\begin{array}{ccc} A & \xrightarrow{p^s} & p^s A \\ \cup & & \cup \\ (p) & \xrightarrow{\sim} & p^{s+1}A \end{array}$$ is a module isomorphism b/c A is domain

$\rightsquigarrow p^sA/p^{s+1}A \xleftarrow[\sim]{p^s\cdot ?} A/(p)$ as vector spaces over the field $A/(p) \Rightarrow d_{p,s}(A) = 1$.

ii) $A/(p^t) =: M'$; if $s \geq t \Rightarrow p^s M' = \{0\} \Rightarrow d_{p,s}(M') = 0$

if $s < t \Leftrightarrow (p^s) \supsetneq (p^t)$ so

$p^sM'/p^{s+1}M' \simeq p^sA/p^{s+1}A$ as $A/(p)$-modules; $d_{p,s}(M') = 1$ by i)

iii) $M'' := A/(q^t)$ but q,p are coprime so $(q^t)+(p) = A \Rightarrow$

$pM'' = \{pa + (q^t) \mid a \in A\} = ((p)+(q^t))/(p) = A/(p) = M'' \Rightarrow p^sM'' = p^{s+1}M''$

$\Rightarrow p^sM''/p^{s+1}M'' = \{0\}$

Summing the contributions from the summands (0 or 1) together, we arrive at the claim of the theorem. $\square$

2) Localization

We've seen a bunch of constructions of rings:
- direct products
- rings of polynomials
- quotient rings
- completions (HW 1)

Now we discuss another construction w. rings – localization. It generalizes the construction of $\mathbb{Q}$ from $\mathbb{Z}$ and amounts to formally inverting elements from suitable subsets in a commutative ring.

2.1) Multiplicative subsets

We start by explaining what kind of subsets we need. Here & below A is a commutative ring.

Definition: A subset $S \subset A$ is multiplicative if
- $1 \in S$ &
- $s, t \in S \Rightarrow st \in S$

Examples (of multiplicative subsets)

1) All invertible elements of A.
2) All non-zero divisors of A.
3) For $f \in A$, $S := \{f^n \mid n \geqslant 0\}$. More generally for $f_1, \ldots f_k \in A$, can take $S := \{f_1^{n_1} \ldots f_k^{n_k} \mid n_1, \ldots n_k \geqslant 0\}$.

4) If $\mathfrak{p}$ is a prime ideal ($ab\in\mathfrak{p} \Rightarrow a\in\mathfrak{p}$ or $b\in\mathfrak{p}$), then $S:=A\setminus\mathfrak{p}$ is multiplicative.

2.2) Construction of localization.

Now we proceed to constructing the localization, $A[S^{-1}]$. Consider $A\times S$ (product of sets), equip it w. equivalence relation $\sim$ defined by

$$(*)\ (a,s)\sim(b,t) \overset{def}{\Longleftrightarrow} \exists\, u\in S \mid uta=usb.$$

$\sim$ is indeed an equivalence relation: the only nontrivial thing is transitivity:

$$\begin{cases}(a_1,s_1)\sim(a_2,s_2) \Rightarrow us_1a_2=us_2a_1 \\ (a_2,s_2)\sim(a_3,s_3)\Rightarrow u's_3a_2=u's_2a_3\end{cases} \Longrightarrow (uu's_2)s_1a_3=uu's_3s_1a_2=uu's_3s_2a_1 = (uu's_2)s_3a_1 \Rightarrow (a_1,s_1)\sim(a_3,s_3)$$

Let $A[S^{-1}]$ be the set of equivalence classes. The class of (a,s) will be denoted by $\frac{a}{s}$.

Addition & multiplication in $A[S^{-1}]$ are given by usual formulas

$$\frac{a_1}{s_1}+\frac{a_2}{s_2}:=\frac{s_2a_1+s_1a_2}{s_1s_2},\quad \frac{a_1}{s_1}\,\frac{a_2}{s_2}:=\frac{a_1a_2}{s_1s_2}$$

Proposition: These operations are well-defined (the result depends only on $\frac{a_1}{s_1},\frac{a_2}{s_2}$, not on $(a_1,s_1),(a_2,s_2)$) & equip $A[S^{-1}]$ w. structure of a commutative ring (w. zero $\frac{0}{1}$ & unit $\frac{1}{1}$). Moreover, $\iota: A\to A[S^{-1}], a\mapsto\frac{a}{1}$, is a ring homomorphism.

Proof: omitted in order not to make everybody very bored...

Def'n: The ring $A[S^{-1}]$ is called the localization of A (w.r.t. S). We view it as an A-algebra via ι.

Remarks: 1) If $0\in S$, then $\frac{a}{s}\sim\frac{0}{1}$ (take $u=0$ in $(*)$) $\forall\, a\in A, s\in S \Rightarrow A[S^{-1}]=\{0\}$. Conversely, if $0\notin S$, then $(1,1)\not\sim(0,1)\Rightarrow A[S^{-1}]\neq\{0\}$.

2) $\ker\iota=\{a\in A \mid \frac{a}{1}\sim\frac{0}{1}\Leftrightarrow \exists\, u\in S \mid ua=0\}$

3) The elements $\frac{a}{1}=\iota(a)$ & $\frac{1}{s}$ generate $A[S^{-1}]$ as a ring.

4) If S consists of invertible elements, then $\iota: A\to A[S^{-1}]$ is injective by 2) & surjective by 3): $\frac{1}{s}=\frac{s^{-1}}{1}$, hence is an isomorphism.

Example/exercise: Let $A=\mathbb{Z}/6\mathbb{Z}$ & $S=\{1,2,4\}$. Then $\ker\iota=\{0,3\}$ & ι is surjective as $\iota(-1)=\frac{2}{-2}=\frac{2}{4}=\frac{1}{2}$, so $A[S^{-1}]\simeq\mathbb{Z}/3\mathbb{Z}$.

2.3) Case when S consists of non-zero divisors

Here the description of $\sim$ simplifies: $(a,s)\sim(b,t)\Leftrightarrow ta=sb$. Also ι is injective. More generally, let $\tilde{S}=\{\text{all non-zero divisors}\}$ so that $S\subset\tilde{S}$. The simplified description of $\sim$ shows that the equivalence on $A\times S$ is the restriction of the equivalence of $A\times\tilde{S}$ hence $A[S^{-1}]$ is naturally a subring of $A[\tilde{S}^{-1}]$ (of all elements $\frac{a}{s}$ w. $s\in S$)

Assume A is a domain $\Rightarrow \tilde{S} = A \setminus \{0\}$. Then every nonzero element of $A[\tilde{S}^{-1}]$ is invertible: $(\frac{a}{b})^{-1} = \frac{b}{a}$ so $A[\tilde{S}^{-1}]$ is a field called the fraction field of A and denoted by $\mathrm{Frac}(A)$. For example, for $A = \mathbb{Z}$ we recover $\mathrm{Frac}(A) = \mathbb{Q}$ and then for general S (not containing 0), $A[S^{-1}]$ is the subring of $\mathbb{Q}$ consisting of all rational numbers w. denominator in S.

Example: Let B be a commutative ring, $A = B[x]$ & $S = \{x^n \mid n \geq 0\}$. Clearly, x^n are non-zero divisors. So the equivalence relation on $A \times S$ is $(f, x^n) \sim (g, x^m) \Leftrightarrow x^m f = x^n g$. An equivalence class (f, x^n) can be viewed as a Laurent polynomial $x^{-n} f$ and ring structures of $A[S^{-1}]$ & $B[x^{\pm 1}]$ match so that $A[S^{-1}] = B[x^{\pm 1}]$.

2.4) Universal property of localization

Let A be a commutative ring & $S \subset A$ be multiplicative $\rightsquigarrow$ ring homomorphism $\iota: A \to A[S^{-1}]$. Note that $\forall s \in S \Rightarrow \iota(s) \in A[S^{-1}]$ is invertible (w. inverse $\frac{1}{s}$)

Proposition: Let $\varphi: A \to B$ be a ring homomorphism s.t. $\varphi(s) \in B$ is invertible $\forall s \in S$. Then the following hold:

1) $\exists!$ ring homom'm $\varphi': A[S^{-1}] \to B$ that makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[d, "\iota"'] \arrow[dr, "\varphi"] & \\ A[S^{-1}] \arrow[r, dashed, "\varphi'"] & B \end{tikzcd}$$

2) φ' is given by $\varphi'\left(\frac{a}{s}\right) = \varphi(a)\varphi(s)^{-1}$.

Sketch of proof:

Existence: need to show that formula in 2) indeed gives a well-defined ring homomorphism.

- φ' is well-defined: WTS $\frac{a}{s} = \frac{b}{t} \Rightarrow \varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$.

Indeed: $\frac{a}{s} = \frac{b}{t} \Longleftrightarrow \exists\, u \in S$ s.t $uta = usb \Rightarrow \varphi(u)\varphi(t)\varphi(a) = \varphi(u)\varphi(s)\varphi(b)$. But $\varphi(u), \varphi(t), \varphi(s)$ are invertible. It follows that $\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$. So φ' is well-defined.

Exercise – on addition & multiplication of fractions. Check that φ' is a ring homomorphism.

Note that φ' makes the diagram in 1) commutative.

Uniqueness: φ' makes diagram comm've $\Longleftrightarrow \varphi'\left(\frac{a}{1}\right) = \varphi(a)\ \forall a \in A$
$\Rightarrow \varphi'\left(\frac{s}{1}\right) = \varphi(s)$ – invertible $\Rightarrow \varphi'\left(\frac{1}{s}\right) = \varphi(s)^{-1} \Rightarrow$
$\varphi'\left(\frac{a}{s}\right) = \varphi'\left(\frac{a}{1}\right)\varphi'\left(\frac{1}{s}\right) = \varphi(a)\varphi(s)^{-1}$. $\square$

We'll discuss applications of this proposition to computing localizations in the next lecture

Remark: One can strengthen the statement as follows: the maps

$$\begin{tikzcd}
\varphi' \in \{\text{ring homomorphisms } \psi : A[S^{-1}] \longrightarrow B\} \ni \psi\circ\iota & \psi \arrow[l, mapsto] \arrow[d, phantom, sloped, "\in"] \\
\varphi \in \{\text{ring homomorphisms } \varphi : A \to B \mid \varphi(s) \in B \text{ is invertible } \forall\, s \in S\} \arrow[u, mapsto] & {}
\end{tikzcd}$$

are mutually inverse. The proof is an exercise.