Lecture 9: Localization of rings & modules, II

1) Localization of rings, cont'd
2) Localization of modules.

Refs: [AM], Sec 3 (up to "Local properties")

1) Localization of rings, cont'd

1.1) Localizations $A[S^{-1}]$ for $S=\{f^n \mid n\geqslant 0\}$

These are usually denoted by $A[f^{-1}]$. The following lemma gives an alternative description.

Lemma: We have ring isomorphism $A[f^{-1}] \simeq A[x]/(xf-1)$

Proof:

We write I for $(xf-1)$. We produce homomorphisms between $A[f^{-1}]$ & $A[x]/I$ using universal properties & show they are mutually inverse.

Let φ be the composition $A \hookrightarrow A[x] \twoheadrightarrow A[x]/I$. Then $\varphi(f)=f+I$ is invertible w. inverse $x+I$. By universal property of localizations (Sec 2.4 of Lec 8) $\exists!$ homomorphism $\varphi': A[f^{-1}] \longrightarrow A[x]/I$ s.t $\varphi'(\frac{a}{1})=\varphi(a)$, it sends $\frac{1}{f}$ to $\varphi(f)^{-1}=x+I$.

On the other hand, consider the homomorphism

$$A[x] \longrightarrow A[f^{-1}],\ F(x)\mapsto F(\tfrac{1}{f}).$$

It sends $xf-1$ to $\frac{1}{f}f-1=0$ hence uniquely factors through

$$\varphi'': A[x]/(xf-1) \longrightarrow A[f^{-1}],\ F(x)+I \mapsto F(\tfrac{1}{f})$$

Now we show that φ' and φ'' are mutually inverse. By Rem. 3) in Sec 2.1 of Lec 8 $A[f^{-1}]$ is generated (as a ring) by the elements $\frac{a}{1}$ & $\frac{1}{f}$ (b/c $\frac{1}{f^n}=(\frac{1}{f})^n$) while $A[x]+I$ is generated by the elements $a+I$ & $x+I$. By the construction, the homomorphisms φ' & φ'' are mutually inverse on the generators (e.g. $\varphi'(\frac{1}{f})=x+I$ & $\varphi''(x+I)=\frac{1}{f}$) so they are mutually inverse. $\square$

Example: Let $A=\mathbb{C}[y,z]/(yz)$ & $f=z+(yz)$. By Lemma,
$A[f^{-1}]\simeq A[x]/(xf-1)\simeq \mathbb{C}[x,y,z]/(yz,xz-1)=[y=x\cdot yz-y\cdot(xz-1)\Rightarrow (yz,xz-1)=(y,yz,xz-1)=(y,xz-1)]=\mathbb{C}[x,y,z]/(y,xz-1)\simeq\mathbb{C}[x,z]/(xz-1)\simeq\mathbb{C}[z][z^{-1}]\simeq\mathbb{C}[z^{\pm1}]$.

Exercise: Let $f_1,\dots,f_k\in A$ & $S=\{f_1^{n_1}\dots f_k^{n_k}\mid n_i\geqslant 0\}$. Then we have a ring isomorphism $A[S^{-1}]=A[(f_1\dots f_k)^{-1}]$ (hint: universal property).

1.2) Localizations $A[S^{-1}]$ for $S=A\setminus\mathfrak{p}$

The remaining multiplicative subset mentioned in Sec 2.1 of Lec 8 is $S:=A\setminus\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This case is very important in the theory and has its own notation: we write $A_\mathfrak{p}:=A[(A\setminus\mathfrak{p})^{-1}]$, cf. Problem 6 of HW2. We'll continue to discuss this case later in the course.

2) Localization of modules.

2.1) Definition

Let A be a commutative ring & $S\subset A$ be a multiplicative subset. Let M be an A-module. Define the localization $M[S^{-1}]$ as the set of equivalence classes $M\times S/\sim$ w. $\sim$ defined by:

(*) $(m,s)\sim(n,t) \overset{def}{\Longleftrightarrow} \exists\, u\in S \mid utm=usn$

Equiv. class of (m,s) will be denoted by $\frac{m}{s}$.

Proposition: $M[S^{-1}]$ has a natural $A[S^{-1}]$-module str're (w. addition of fractions) & $A[S^{-1}]\times M[S^{-1}]\to M[S^{-1}]$ given by $\frac{a}{s}\frac{m}{t}:=\frac{am}{st}$.

Proof: for the same price as the ring structure on $A[S^{-1}]$. $\square$

Remark: Localizing the regular module A, we get the regular module $A[S^{-1}]$.

2.2) Basic properties of $M[S^{-1}]$

The ring homomorphism $\iota: A\to A[S^{-1}]$ gives an A-module structure on $M[S^{-1}]$: $a\frac{m}{s}=\frac{am}{s}$. The map $M\xrightarrow{\iota_M} M[S^{-1}]$, $m\mapsto\frac{m}{1}$, is A-module homomorphism ($\iota=\iota_A: A\to A[S^{-1}]$ is a special case).

The next result is analogous to Proposition in Sec 2.4 of Lec 8.

Proposition:

1) $\ker\iota_M=\{m\in M \mid \exists\, u\in S \text{ s.t. } um=0\}$. In particular, ι is injective iff $um=0\Rightarrow m=0$ (S acts by non-zero divisors on M).

2) $\operatorname{im}\iota_M$ generates $M[S^{-1}]$ as $A[S^{-1}]$-module. So, $M[S^{-1}]=\{0\}\Leftrightarrow$ $\iota_M=0 \Leftrightarrow \ker\iota_M=M \Leftrightarrow$ [1)] $\forall\, m\in M\ \exists\, u\in S$ w. $um=0$.

3) Universal property of ι_M: let N be $A[S^{-1}]$-module and $\xi \in \mathrm{Hom}_A(M,N)$. Then $\exists!\ \xi' \in \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}],N)$ making the following diagram commutative:

$$\begin{tikzcd} M \arrow[d, "\iota_M"'] \arrow[dr, "\xi"] & \\ M[S^{-1}] \arrow[r, dashed] & N \end{tikzcd}$$

We have $\xi'(\frac{m}{s}) = \frac{1}{s}\xi(m)$.

4) The maps $\xi \mapsto \xi'$ & $\psi \mapsto \psi\circ\iota_M$ are mutually inverse $\mathrm{Hom}_A(M,N) \leftrightarrows \mathrm{Hom}_{A[S^{-1}]}(M[S^{-1}],N)$.

Sketch of proof:

1) exercise (cf. remark 2) in Sec 2.1 of Lec 8).

2) $\mathrm{Span}_{A[S^{-1}]}\, \mathrm{im}\, \iota_M \ni \frac{1}{s}\cdot\frac{m}{1} = \frac{m}{s}$ so coincides w. $M[S^{-1}]$. The remaining claims in 2) follow.

3) This is similar to the universality property for localizations of rings, details are an exercise.

4) We need to show $\xi'\circ\iota_M = \xi$ & $(\psi\circ\iota_M)' = \psi$. The former follows from the commutative diagram in 3). The latter follows b/c for $\xi := \psi\circ\iota_M$ both ξ' & ψ make the diagram in 3) commutative & 3) states that this property determines a homomorphism $M[S^{-1}] \to N$ uniquely. $\square$

We apply 3) to produce, from an A-linear map $\varphi: M_1 \to M_2$, an $A[S^{-1}]$-linear map $\varphi[S^{-1}]: M_1[S^{-1}] \to M_2[S^{-1}]$. Take $\tilde{\varphi}:=\iota_{M_2}\circ\varphi: M_1 \to M_2[S^{-1}]$, and set $\varphi[S^{-1}]:=\tilde{\varphi}'$, explicitly

$$\varphi[S^{-1}]\left(\frac{m_1}{s}\right)=\frac{\varphi(m_1)}{s},\ \forall\, m_1\in M_1,\ s\in S.$$

Important exercise: Check that

a) $id_M[S^{-1}]=id_{M[S^{-1}]}$.

b) For $\varphi_1: M_1\to M_2, \varphi_2: M_2\to M_3$, have $(\varphi_2\circ\varphi_1)[S^{-1}]=\varphi_2[S^{-1}]\circ\varphi_1[S^{-1}]$.

c) For $\varphi,\varphi': M_1\to M_2$, have $(\varphi+\varphi')[S^{-1}]=\varphi[S^{-1}]+\varphi'[S^{-1}]$.

Rem: A lot of results in this section (and below in this entire topic) will be revisited when we discuss category theory in the 2nd part of the class. For example a) & b) will essentially imply that "localization is a functor", c) that it is an "additive functor" and 3) of Proposition means that certain functors are "adjoint."

2.3) Submodules in $M[S^{-1}]$

Let M be an A-module, $N\subset M$ A-submodule. Note that for $m,n\in N$, $s,t\in S$ we have $(m,s)\sim(n,t)$ in $N\times S \Leftrightarrow (m,s)\sim(n,t)$ in $M\times S$. So $N[S^{-1}]$ can be viewed as a subset in $M[S^{-1}]$, in fact, it's an $A[S^{-1}]$-submodule (exercise).

Recall that the localization of the regular A-module A is the regular $A[S^{-1}]$-module $A[S^{-1}]$. So, for an ideal $I\subset A$, get an ideal $I[S^{-1}]\subset A[S^{-1}]$.

Exercise 1: Show that for submodules $N_1, N_2 \subset M$ we have $(N_1+N_2)[S^{-1}] = N_1[S^{-1}] + N_2[S^{-1}]$ (hint: common denom'r), and similarly for intersections. Also $N_1 \subset N_2 \Rightarrow N_1[S^{-1}] \subset N_2[S^{-1}]$.

It turns out that every $A[S^{-1}]$-submodule of $M[S^{-1}]$ is of the form $N[S^{-1}]$. Namely, for an $A[S^{-1}]$-submodule $N' \subset M[S^{-1}]$, consider $\iota_M^{-1}(N') \subset M$, this is an A-submodule as the kernel of the A-linear map obtained as composition of ι_M & proj'n $M[S^{-1}] \twoheadrightarrow M[S^{-1}]/N'$

Proposition: The maps $N' \mapsto \iota_M^{-1}(N')$ & $N \mapsto N[S^{-1}]$ are mutually inverse bijections between:

$\{A[S^{-1}]\text{-submodules } N' \subset M[S^{-1}]\}$ &

$\{A\text{-submodules } N \subset M \mid \underbrace{sm \in N \text{ for } s \in S, m \in M \Rightarrow m \in N}_{\text{condition } (\dagger)}\}$

Proof: Step 1: Show that $\iota^{-1}(N')$ satisfies $(\dagger)$:

$sm \in \iota_M^{-1}(N') \Leftrightarrow \iota_M(sm) \in N' \Leftrightarrow \frac{s}{1}\iota_M(m) \in N' \Leftrightarrow \iota_M(m) = \frac{1}{s}\frac{s}{1}\iota_M(m) \in N' \Leftrightarrow m \in \iota_M^{-1}(N')$.

So we have two maps between the two sets, need to show that they are mutually inverse

Step 2: $\iota_M^{-1}(N[S^{-1}]) = N$ for $\forall$ A-submodule N satisfying $(\dagger)$:

$\iota_M^{-1}(N[S^{-1}]) = \{m \in M \mid \iota_M(m) \in N[S^{-1}] \Leftrightarrow \frac{m}{1} = \frac{n}{s}$ for some $n \in N, s \in S$ $\Leftrightarrow \exists\, u \in S \mid usm = un \Leftrightarrow [un \in N, us \in S\ \&\ (\dagger)]\ m \in N\} = N$.

Step 3: $(\iota_M^{-1}(N'))[S^{-1}] = N'$: $(\iota^{-1}(N'))[S^{-1}] = \{\frac{n}{s} \mid \frac{n}{1} \in N' \Leftrightarrow [\frac{s}{1}$ is invertible$] \Leftrightarrow \frac{n}{s} \in N'\} = N'$. $\square$

Corollary: Suppose M is a Noetherian A-module. Then $M[S^{-1}]$ is a Noetherian $A[S^{-1}]$-module. In particular, if A is a Noetherian ring, then so is $A[S^{-1}]$.

Proof:

Let $N \subset M$ be a submodule so that $N = \mathrm{Span}_A(m_1, \dots m_k)$. Then $N[S^{-1}] = \mathrm{Span}_{A[S^{-1}]}(\frac{m_1}{1}, \dots, \frac{m_k}{1})$ (exercise). By Proposition, every $A[S^{-1}]$-submodule of $M[S^{-1}]$ is of the form $N[S^{-1}]$, hence finitely generated.

Exercise 2: Prove a similar claim for Artinian modules (hint: the bijections in Proposition respect inclusions).