IMPORTANT INFORMATION ON MATH 380

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1. Content of the class

The main subject of the class is Commutative Algebra. Commutative algebra studies *commutative rings* and their *modules*. Commutative rings generalize fields: we drop the axiom of the existence of multiplicative inverse. Classical examples include the ring of integers as well as the rings of polynomials. Modules over rings generalize vector spaces over fields. Unlike vector spaces, modules over an arbitrary ring have an intricate and complicated structure.

Commutative algebra emerged in the second half of the 19th century. Two primary origins of the subject were Number theory and Invariant theory. I refer you to the first chapter in [E] for a more thorough discussions of the origins of Commutative algebra. Nowadays, Commutative algebra is related to several other parts of Mathematics, including algebraic Number theory and Algebraic geometry. I do not plan to discuss connections with Number theory extensively (for example, because there is a separate class on algebraic Number theory offered in Fall 2020) but I do plan to give a brief introduction to Algebraic geometry and discuss some developments of Commutative algebra in that context.

Another topic to be covered in this course is a brief introduction to Category theory, which is needed for a meaningful discussion of several constructions with modules over commutative rings.

Here is a more detailed description of what we are going to cover. All the terminology that appears in this description without explanation will be explained in the course.

1.1. **Examples and classes of rings, ideals and modules.** The goal here is to introduce a basic terminology related to rings, their ideals and modules; describe important classes of these objects (such as, say, projective modules) and give examples.

1.2. **Principal ideal domains (PID) and their modules.** PID's is an important yet easy class of rings that includes the ring of integers and the rings of polynomials in a single variable with coefficients in a field. The formal definition of a PID is that it is a domain (no zero divisors) and every ideal is principal, i.e., is generated by a single element. A remarkable feature of this class of rings is that it is easy to describe finitely generated modules over them, this description generalizes the classification of finitely generated abelian groups (=modules over $\mathbb{Z}$).

1.3. **Noetherian and Artinian rings and modules.** A ring is *Noetherian* if every ideal is generated by finitely many elements. These rings are ubiquitous in Commutative algebra, in fact, most rings we encounter are Noetherian. Being Noetherian is equivalent to the condition that every ascending chain of ideals terminates. We will establish basic properties and introduce basic classes of Noetherian rings and discuss modules over these rings. We will also discuss another class, *Artinian rings*. These are defined by the condition that every descending chain of ideals terminates.

1.4. **Constructions with rings (and modules).** We are going to discuss two constructions that allow to produce new rings (and sometimes modules) from existing ones. The first construction is the *localization*, where we invert certain elements in a ring. For example, the rational numbers are obtained from the integers by localization. The second construction is that of integral extension/ integral closure. For example, the rings of algebraic integers that are of paramount importance for algebraic Number theory are obtained via taking the integral closure of $\mathbb{Z}$ in finite field extensions of $\mathbb{Q}$. We also plan to discuss basic results on integral extensions such as the Noether normalization theorem.

1.5. **Connections to Algebraic geometry.** Here we discuss basic connections between Commutative algebra and Algebraic geometry. This allows to put several classical results and constructions of Commutative algebra (such as the primary decomposition theorem into a generic framework). We introduce closed subvarieties in the affine spaces and discuss their connections with the ideals in the ring of polynomials. We also discuss irreducible subvarieties and irreducible components. We give a geometric interpretation of localization. We will also study local rings in more detail and prove the Nakayma lemma. Finally, we discuss a geometric significance of projective modules.

1.6. **Introduction to Category theory.** Category theory provides a unifying language for various constructions that appear throughout Mathematics. We will introduce the notions of categories, functors and functor morphisms and give examples. We will prove the Ioneda lemma and discuss representable functors and adjointness. Then we will concentrate on the categories of modules and functors between them. We will introduce the notions of left and right exact functors and give a categorical interpretation of projective modules.

1.7. **Tensor products and rings and modules.** The final topic of the class is tensor products of rings and modules. We start by defining tensor product of modules and explain how to compute them. We then study tensor products from the perspective of Category theory. Next, we discuss tensor products of rings and their categorical and geometric significance. Time permitting, we will discuss tensor and symmetric algebras of a module.

2. Homeworks

There will be five homeworks, the due dates are on the course webpage. Each homework is worth 12% of the total, i.e., 24 points. You should expect that the total number of points in a homework is bigger than the maximal score. In particular, you don't need to solve all problems correctly to get the maximal score. However, you strongly encouraged to try to solve everything. Some of the homework problems may be used to prove statements in class. And some are going to be on recurring topics.

You should expect a significant variety of homework problems. Some of them will be routine exercise, while some others will be difficult and will require new ideas. Some problems will be closely related to what we cover in class, while some others will concentrate on important topics that will not be covered in class due to lack of time.

I don't plan to post complete solutions. Instead, after each homework I'll poll you and ask your preferences for problems whose solutions you want to see.

3. References

There is no required textbook, but there are three recommended textbooks:

– [AM] is a classical introductory textbook in Commutative algebra. It is clearly written and concise.

– [E] is a very comprehensive introduction to the subject. It's more modern than [AM] and also emphasizes connections to Algebraic geometry. It's also more advanced.

– [R] is a recent introductory textbook on Category theory.

4. Lectures and office hours

Both lectures and office hours are going to be via zoom – at least for time being... Videos of lectures will be posted on Canvas and lecture notes will be posted both on Canvas and on the course webpage. From time to time, posted lecture notes are going to contain some additional material.

Generally, there will be two office hours in a week. Times are to be determined via collecting preferences.

5. Questions and inquiries

Please do not hesitate to contact me if you have questions! I'll try to chat with every student taking the class early in the semester and learn about your background.

References

[AM] M. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[E] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, GTM 150, Springer-Verlag, 2004.

[R] E. Riehl, *Category theory in context.* Available online.