IMPORTANT INFORMATION ON MATH 380, FALL 2025

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Contents

1. Introduction

The main subject of the class is Commutative Algebra. Commutative algebra mostly studies *commutative rings/algebras* and their *modules.* Commutative rings generalize fields: we drop the axiom that the multiplicative inverse exists. Classical examples include the ring of integers as well as the rings of polynomials. Modules over rings generalize vector spaces over fields. Unlike vector spaces, modules over a general ring have an intricate and complicated structure.

Commutative algebra emerged in the second half of the 19th century. Two primary origins of the subject were Number theory and Invariant theory. I refer you to the first chapter in [E] for a more thorough discussions of the origins of Commutative algebra. Nowadays, Commutative algebra is related to several other parts of Mathematics, including Algebraic Number theory and Algebraic geometry. We will spend some time discussing these connections.

The second topic to be covered in this course is a brief introduction to Category theory, which, for the purposes of this class, is needed for a meaningful discussion of several classes of/ constructions with modules over commutative rings but is of paramount importance for Mathematics in general.

2. Content of the class

Below I provide a more detailed – and technical– description of what we are going to cover. The class naturally splits into two large and one smaller sections:

(1) Basics of Commutative algebra.

(2) Results/constructions of relevance for Algebraic geometry (Nullstellensatz and some consequences) and Number theory (Dedekind domains and factorization of ideals).
(3) Category theory with connections to Commutative algebra.

All the terminology that appears in the description below without explanation will be explained in the course.

2.1. **Basics of Commutative algebra.** The main players in Commutative Algebra are rings (and their homomorphisms), ideals, and modules. The goal of this section of the class is to explain basic constructions and examples of these objects and study their basic properties. Here is what we are going to cover in more detail.

2.1.1. *Examples and classes of rings, ideals and modules.* The goal here is to introduce a basic terminology related to rings and algebras, their ideals and modules; describe important classes of these objects (such as, say, prime ideals and free modules) and give examples.

2.1.2. *Noetherian and Artinian rings and modules.* A ring is *Noetherian* if every ideal is generated by finitely many elements. These rings are ubiquitous in Commutative algebra, in fact, most rings we encounter are Noetherian. Being Noetherian is equivalent to the condition that every ascending chain of ideals terminates. We will establish basic properties and introduce basic classes of Noetherian rings and discuss modules over these rings. We will also briefly discuss another class, *Artinian rings.* These are defined by the condition that every descending chain of ideals terminates.

2.1.3. *Principal ideal domains (PID) and their modules.* PID's is an important yet easy class of rings that includes the ring of integers and the rings of polynomials in a single variable with coefficients in a field. The formal definition of a PID is that it is a domain (no zero divisors) and every ideal is principal, i.e., is generated by a single element. A remarkable feature of this class of rings is that it is easy to describe finitely generated modules over them, this description generalizes the classification of finitely generated abelian groups (=modules over $\mathbb{Z}$).

2.1.4. *Integral extensions.* Taking integral extension is an important construction of rings, this is a ring counterpart of algebraic extensions of fields studied in MATH 370. We will explain basics on integral extensions and prove the Noether normalization theorem: every finitely generated algebra over a field that is a domain is an integral extension of the algebra of polynomials.

2.1.5. *Localization of rings and modules.* To localize means to invert some elements. This gives another important construction of rings. For example, the rational numbers are obtained from the integers by localizing all nonzero elements. Localization can also be applied to modules giving an example of a *functor.* We will explain construction and basic properties of localization for rings and modules.

2.2. **Connections with Algebraic geometry and Number theory.**

2.2.1. *Connections to Algebraic geometry.* Algebraic geometry starts with studying geometric properties of sets of solutions to systems of polynomial equations. Its interactions with Commutative algebra that can be traced back to the work of Hilbert in the end the 19th century are numerous and beneficial for both fields: oversimplifying, one can say that Commutative algebra provides a foundation for Algebraic geometry, while Algebraic geometry provides a geometric intuition for otherwise abstract Commutative algebra constructions. To illustrate this connection, we will prove Hilbert's Nullstellensatz (translated as "Theorem on location of zeroes") and establish a bijection between radical ideals in the algebra of polynomials in n variables over an algebraically closed field and the algebraic subsets in the n-dimensional space.

2.2.2. *Connections to Number theory.* Connections to Number theory goes back at least to the work of Dedekind (around 1870). Similarly to algebraic closures of fields, one can talk about integral closures of rings. For example, the rings of algebraic integers that are of paramount importance for algebraic Number theory are obtained via taking the integral closure of $\mathbb{Z}$ in finite field extensions of $\mathbb{Q}$. It was understood that the unique factorization in rings of algebraic integers is important for Number theory – for example, for the proof of the Fermat last theorem. Dedekind realized, that while the unique factorization of elements may fail, it still holds on the level of ideals. We plan to discuss Dedekind domains (generalizations of the rings of algebraic integers) and the uniqueness of factorizations of ideals into the product of prime ideals.

2.3. **Category theory with connections to Commutative algebra.** Category theory is a common language of modern Mathematics. Various constructions and questions of interest in Commutative algebra are categorical in nature. The goal of this part of the course is to introduce basic notions of category theory and apply them to the study of tensor product and Hom modules over commutative rings.

2.3.1. *Categories, functors, functor morphisms.* These are the basic objects that Category theory studies. We will introduce these notions and give examples. Then we will prove the Yoneda Lemma that is a foundational result about functor morphisms.

2.3.2. *Representing objects with application to tensor products.* Many constructions in Mathematics arise as "objects representing functors", which makes sense thanks to the Yoneda Lemma. We will apply this to revisit/ introduce various notions of products in Commutative algebra, most importantly, tensor products of modules and rings.

2.3.3. *Adjoint functors and tensor-Hom adjunction.* In Linear algebra and beyond we talk about adjoint linear maps (or operators). Similarly, it makes sense to speak about two functors being adjoint to each other. This is a very important relationship giving rise to various universal constructions in Algebra. We will discuss adjoint functors and give several examples, including the adjunction between tensor product and Hom functors.

2.3.4. *Additive functors, exactness of functors, projective and flat modules.* Our main focus is on functors between categories of modules over rings. Additive functors are those which have a certain compatibility with the module structure. We will define and give examples of additive functors. The main property of these functors we care about in Commutative algebra is their *exactness*, i.e., how they behave on exact sequences of modules. We will study the exactness of tensor product and Hom functors. This, in particular, will allow us to define two important and interesting classes of modules:

projective and flat modules. We will then discuss a connection between projective and (locally) free modules. In particular, we introduce local rings, prove a basic yet very important result about their modules, the Nakayama lemma, and show that reasonable projective modules over local rings are free.

3. Prerequisites

There are two official prerequisites for this class: MATH 350 (Introduction to Abstract Algebra) and MATH 370 (Fields and Galois theory). The former covers finite groups and, closer to the end, commutative rings and ideals. Abelian groups and their classification, rings and ideals, unique factorization property, Euclidian rings, etc. studied in MATH 350 are directly related to what we are going to study in our class. MATH 370 covers field extensions and Galois theory. Fields, rings of polynomials and their quotients also appear often in our class. More advanced things such as finite and algebraic extensions of fields, algebraic closures etc. appear somewhat in the background in our study of integral extensions and closures of rings. And various things related to MATH 370 may appear in homeworks (see the next section for more about homeworks). You can check Asher Auel's webpage at Dartmouth for syllabi, references and homeworks for MATH 350 (e.g., Fall 2017) and MATH 370 (e.g., Spring 2019).

You will also benefit from a good knowledge of Linear Algebra (and not just for this class). E.g. a good understanding of the Jordan normal form theorem and of tensor products of vector spaces will be helpful. At Yale, this is studied in MATH 340.

Even more importantly, previous Algebra classes give you an exposure to abstract algebraic considerations. This is important for understanding the material in our class. To put this somewhat differently, some mathematical maturity will help you to benefit from our class.

4. Homeworks

There will be five homeworks, the due dates to be announced. Each homework is worth 14% of the total, i.e., 14 points. You should expect that the total number of points in a homework is bigger than the maximal score. In particular, you don't need to solve all problems correctly to get the maximal score. However, you strongly encouraged to try to solve everything. Some of the homework problems may be used to prove statements in class. And some are going to be on recurring topics.

You should expect a significant variety of homework problems. Some of them will be routine exercise, while some others will be difficult and will require new ideas. Some problems will be closely related to what we cover in class, while some others will concentrate on important topics that will not be covered in class due to lack of time.

You should also expect that solving the homework will require a significant time and effort investment. Students generally find that this class is quite challenging...

I don't plan to post complete solutions. Instead, after each homework I'll poll you and ask your preferences for problems whose solutions you want to see.

5. References

There is no required textbook, but there are five textbooks that may be useful. One thing to keep in mind is that books on Category theory are generally require a more extensive background than what is expected for this class. The same is true for books on Algebraic geometry.

– [AM] is a classical introductory textbook in Commutative algebra. It is clearly written and concise. This is going to be our main reference for the basics of Commutative algebra section.

– [E] is a very comprehensive introduction to Commutative algebra. It's more modern than [AM] and also emphasizes connections to Algebraic geometry. It's also more advanced.

– [HS] is a classical textbook on Homological algebra and Category theory.

– [R] is a recent introductory textbook on Category theory.

– [V] is a general book on Algebra aimed at undergraduate students. It's good for background reading. We will also follow the exposition in this book for some of our topics.

– [N] is a book on Algebraic Number theory, that we are going to use for our lectures on the subject.

6. Lectures and office hours

Lecture notes will be posted on the course webpage. From time to time, posted lecture notes are going to contain some additional material.

Generally, there will be two office hours per week. Times are to be determined via collecting preferences early in the semester.

7. Questions and inquiries

Please do not hesitate to contact me if you have questions!

References

[AM] M. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[E] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, GTM 150, Springer-Verlag, 2004.

[HS] P.J. Hilton, U. Stammbach, *A Course in Homological Algebra*, 2nd edition. Springer, 2012.

[N] J. Neukirch, *Algebraic Number theory.* Grundlehren der mathematischen Wissenschaften 322, Springer. 1999.

[R] E. Riehl, *Category theory in context.* Available online.

[V] E. Vinberg, *A Course in Algebra.* GSM 56, Springer Verlag, 2003.