MATH 380, HOMEWORK 1, DUE SEPT 21

There are 9 problems worth 30 points total. Your score for this homework is the minimum of the sum of the points you've got and 24. Note that if the problem has several related parts, such as Problem 8, you can use previous parts to prove subsequent ones and get the corresponding credit. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative and contain 1.

Problem 1, 5pts total. Let $\varphi : A \to B$ be a ring homomorphism and let J be an ideal in B. Set $I := \varphi^{-1}(J)$.

a, 1pt) Prove I is an ideal in A.

b, 1pt) Let J be prime. Is it always true that I is prime?

c, 1pt) Let J be maximal. Is it always true that I is maximal?

d, 1pt) Is it always true that $B\varphi(I) \subset J$?

e, 1pt) Is it always true that $J \subset B\varphi(I)$?

If you think a statement is true, provide a proof. If you think it is false, provide a counterexample.

Problem 2, 5pts total. Consider the ring $A = \mathbb{Z}[\sqrt{-5}](:= \mathbb{Z}[x]/(x^2+5))$, its elements can be thought as expressions $a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$ and added and multiplied accordingly. *This ring provides an example of a quadratic extension of* $\mathbb{Z}$ *that is not a UFD. The relevant decomposition is* $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. *One of the things that the problem indicates is how this failure gets fixed if instead of elements of the ring we consider nonzero ideals.*

Consider the following ideals of A

$$I_1 := (2), I_2 := (3), I_3 := (1+\sqrt{-5}), I_4 := (1-\sqrt{-5}),$$
$$I_{13} := I_1 + I_3, I_{23} := I_2 + I_3, I_{14} = I_1 + I_4, I_{24} := I_2 + I_4.$$

a, 2pt) Describe the quotients of A by the ideals $I_{13}, I_{23}, I_{14}, I_{23}, I_1, I_2, I_3, I_4$. *Hint/the form of the answer – in almost all cases your answer should be the direct sum of residue rings, i.e., the rings of the form* $\mathbb{Z}/n\mathbb{Z}$, *while in one case you should get a quotient ring of the ring of polynomials with coefficients in* $\mathbb{Z}/2\mathbb{Z}$.

b, 1pt) Prove that $I_1 = I_{13}I_{14}, I_2 = I_{23}I_{24}, I_3 = I_{13}I_{23}, I_4 = I_{14}I_{24}$.

c, 1pt) Which of the eight ideals $I_1, I_2, I_3, I_4, I_{13}, I_{23}, I_{14}, I_{24}$ are maximal? Which are prime?

d, 1pt) Which of these ideals coincide with their radicals?

Problem 3, 4pts total. *This problem deals with an important construction of rings known as the completion.* Let A be a ring and I be its ideal. Consider the direct product $\prod_{k=1}^{\infty} A/I^k$.

a, 1pt) Consider the subset of $\prod_{k=1}^{\infty} A/I^k$ consisting of all collections $(a_k)_{k=1}^{\infty}$ with $a_k \in A/I^k$ such that a_k coincides with a_{k+1} modulo I^k for all k. Show that this subset is a subring of $\prod_{k=1}^{\infty} A/I^k$. We denote this subring by $\hat{A}$. *An important example: when $A = \mathbb{Z}$ and $I = (p)$, where p is prime, the ring $\hat{A}$ is the ring of p-adic integers, it plays an important role in Algebraic Number theory and is likely to be discussed in that class.*

b, 1pt) *We will concentrate on another important example: the ring of formal power series.* Let $A = B[x]$, where B is another ring, and $I = (x)$. Show that an element of $\hat{A}$ can be thought as a "formal power series", an infinite sum $\sum_{i=0}^{\infty} b_i x^i$, where $b_i \in B$ (unlike with polynomials, we do not require that the sum is finite). Write the addition and multiplication of two formal power series $\sum_{i=0}^{\infty} a_i x^i$ and $\sum_{i=0}^{\infty} b_i x^i$. *For $B = \mathbb{R}$ or $\mathbb{C}$, power series should be familiar from Calculus or Real/Complex Analysis. Unlike there, we do not require our power series to converge anywhere – which is why they are called formal.* The common notation for the ring of formal power series $\hat{A}$ is $B[[x]]$.

c, 1pt) *The ring of formal power series is closely related to the ring of polynomials. But it behaves differently, in fact, in many respects, simpler. The same applies to the p-adic integers vs the integers. In this part we discuss invertible elements in $B[[x]]$.* Prove that $\sum_{i=0}^{\infty} b_i x^i$ is invertible in $B[[x]]$ if and only if b_0 is invertible in B.

d, 1pt) Let B be a field. Describe all possible ideals in $B[[x]]$.

Problem 4, 2pts total. Let A be a ring, M be an A-module, and $m \in M$.

1, 1pt) We define the subset $\mathrm{Ann}_A(m) := \{a \in A | am = 0\}$ ("Ann" stands for the "annihilator"). Prove that $\mathrm{Ann}_A(m)$ is an ideal in A.

2, 1pt) We define the subset $\mathrm{Ann}_A(M) := \{a \in A | am = 0, \forall m \in M\}$. Assume that M is generated by elements $m_1, \ldots, m_k \in M$. Prove that $\mathrm{Ann}_A(M) = \bigcap_{i=1}^{k} \mathrm{Ann}_A(m_i)$.

Problem 5, 2pts. Let A be a domain and I be a nonzero ideal. Suppose that I is free as an A-module. Prove that $I \cong A$, an isomorphism of A-modules (*and not an equality of subsets of* A!).

Problem 6, 2pts. Let M be an A-module with generators $m_1, \ldots, m_k$. Assume that the kernel of the corresponding epimorphism (=surjective homomorphism) $A^{\oplus k} \twoheadrightarrow M$ is generated by the elements $a_i \in A^{\oplus k}, i = 1, \ldots, \ell$, so that $a_i = (a_{i1}, \ldots, a_{ik})$ for $a_{ij} \in A$. Let N be another A-module. Produce an A-module isomorphism between the following two modules:

1) $\mathrm{Hom}_A(M, N)$,

2) and the submodule $\{(n_1, \ldots, n_k) \in N^{\times k} | \sum_{j=1}^{k} a_{ij} n_j = 0, \forall i = 1, \ldots, \ell\}$ of $N^{\times k}$.

Problem 7, 3pts total. Let I be a (possibly, infinite) set and M_i be a collection of A-modules indexed by i. Let N be another A-module.

a, 1pt) Construct a natural A-module isomorphism

$$\mathrm{Hom}_A(\bigoplus_{i\in I} M_i, N) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_A(M_i, N).$$

b, 1pt) Construct a natural A-module isomorphism

$$\mathrm{Hom}_A(N, \prod_{i\in I} M_i) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_A(N, M_i).$$

c, 1pt) Suppose N is finitely generated. Construct a natural isomorphism

$$\mathrm{Hom}_A(N, \bigoplus_{i\in I} M_i) \xrightarrow{\sim} \bigoplus_{i\in I} \mathrm{Hom}_A(N, M_i).$$

Problem 8, 5pts total. *The goal of this problem is to produce an example of a projective module which is not free – and is even weirder than that!* Let $A = \mathbb{Z}[\sqrt{-5}]$. Consider the ideal $I := I_{13}$ from Problem 2 as an A-module. Note that it is generated by two elements, 2 and $1+\sqrt{-5}$, so we have an A-module epimorphism $\pi : A^{\oplus 2} \mapsto I$ given by $(\alpha,\beta) \mapsto 2\alpha + (1+\sqrt{-5})\beta$.

a, 2pts) Identify $\mathrm{Hom}_A(I, A)$ with the A-module M that is realized as follows: we consider the A-module $\mathbb{Q}[\sqrt{-5}]$ and its submodule $M := \{a + b\sqrt{-5} | 2a, 2b, a+b \in \mathbb{Z}\}$.

b, 1pt) Prove that there is an A-module homomorphism $\iota : I \to A^{\oplus 2}$ such that $\pi \circ \iota = \mathrm{id}$.

c, 1pt) Deduce that I is a projective but not free A-module.

d, 1pt) Moreover, identify $\ker \pi$ with I as A-modules. Deduce an isomorphism $I^{\oplus 2} \cong A^{\oplus 2}$. *So $I^{\oplus 2} \cong A^{\oplus 2}$ but $I \not\cong A$. This clearly cannot happen for vector spaces or abelian groups.*

Problem 9, 2pts total. *If we change A and I "slightly", then the conclusions of the previous problem no longer hold.* Now let $A = \mathbb{Z}[x]$. Consider the ideal $I = (2, x)$. We still have the epimorphism $\pi : A^{\oplus 2} \twoheadrightarrow I$ corresponding to the generators $2, x$ of I.

a, 1pt) Establish an isomorphism $\mathrm{Hom}_A(I, A) \cong A$ of A-modules.

b, 1pt) Prove that there is no A-module homomorphism $\iota : I \to A^{\oplus 2}$ such that $\pi \circ \iota = \mathrm{id}$.

This, in particular, implies that I is not a projective A-module – you are not responsible for proving this. So one could ask: what is the difference between the two rings in Problems 8 and 9. One difference is that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, while $\mathbb{Z}[x]$ is. But also these rings have different "dimensions": for $\mathbb{Z}[\sqrt{-5}]$ it is 1, and for $\mathbb{Z}[x]$ it is 2. We will have a bonus discussion of dimensions later in the class.