MATH 380, HOMEWORK 2, DUE OCT 5

There are 10 problems worth 32 points total. Your score for this homework is the minimum of the sum of the points you've got and 24. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homework 1. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative and contain 1.

The first problem is about factorization into irreducibles in Noetherian domains. Its purpose is to illustrate what being Noetherian tells us about the structure of a ring. It also shows that not all of the three equivalent conditions of being Noetherian are created equal – from the point of view of a particular problem – I don't know solutions that would use finite generation of ideals, the AC condition gives a longish solution, while the most elegant solution is produced by using that every set of ideals has a maximal element.

Problem 1, 3pts total. Let A be a domain. We say $a \in A$ is irreducible if, first, it is not invertible, and, second, $a = a_1 a_2$ implies that one of a_i's is invertible. *Do not confuse irreducible and prime elements: being prime is a stronger condition, unless A is a UFD, in which case the two are equivalent.* Prove that if A is Noetherian, then every element a decomposes into the product of irreducible elements and an invertible element.

The next few problems have to do with an important concept of a graded ring. We don't have time to discuss this in class – but graded rings are VERY important so we cannot bypass them completely.

Definition 1. *Let A be a ring. By a grading (or, more precisely, $\mathbb{Z}_{\geqslant 0}$-grading) on A we mean a direct sum decomposition $A = \bigoplus_{i=0}^{\infty} A_i$ of abelian groups such that $A_i A_j \subset A_{i+j}$ for all i and j, and $1 \in A_0$. More generally, if A is a B-algebra, then by an algebra grading on A we mean the direct sum decomposition as above, where all A_i are B-submodules with the same condition on multiplication and the unit. A graded ring (or algebra) is a ring (or algebra) equipped with a grading.*

In particular, A_0 is a subring, and A is an A_0-algebra.

An example: $A = B[x_1, \ldots, x_n]$, where A_i is the span of degree i monomials. This can be generalized by assigning arbitrary positive integer degrees to the variables $x_1, \ldots, x_n$. The following problem gives an easy – but quite general construction of new graded algebras from existing ones. By a homogeneous element in a graded ring (or algebra) A we mean an element of some A_i. By a homogeneous ideal in A we mean an ideal I satisfying $I = \bigoplus_{i=0}^{\infty}(I \cap A_i)$.

Problem 2, 3 points total. Let $\tilde{A}$ be a graded ring and I its homogeneous ideal.

1, 1pt)Equip the quotient $\tilde{A}/I$ with a grading.

2, 1pt) Let $a_1, \ldots, a_k \in \tilde{A}$ be homogeneous elements. Prove that $(a_1, \ldots, a_k)$ is a homogeneous ideal.

3, 1pt) Assume $\tilde{A}$ is a Noetherian ring. Show that I is generated by finitely many **homogeneous** elements.

One remarkable feature of graded rings is that being Noetherian for such rings is closely related to being finitely generated. This was already observed by Hilbert in his work described in the bonus of Lecture 6.

Problem 3, 3 points total. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded ring. Assume A_0 is Noetherian. Prove that the following three conditions are equivalent.
a) A is Noetherian.
b) As an A_0-algebra, A is generated by a finite collection of homogeneous elements of positive degree.
c) The ideal $A_{>0} := \bigoplus_{i=1}^{\infty} A_i$ in A is generated (as an ideal) by a finite collection of homogeneous elements of positive degree.

Here is one more construction of a graded algebra. Let A be a ring and $I \subset A$ be an ideal. Set $\operatorname{gr}_I A := \bigoplus_{i=0}^{\infty} I^i/I^{i+1}$, where $I^0 := A$. It has a natural abelian group structure and multiplication defined on homogeneous elements by

$$(a + I^{i+1})(b + I^{j+1}) := ab + I^{i+j+1}, a \in I^i, b \in I^j,$$

and then extended by distributivity. So $\operatorname{gr}_I A$ is a graded ring.

Problem 4, 3pts total. Suppose A is Noetherian.
1, 2pts) Prove that $\operatorname{gr}_I A$ is a finitely generated A/I-algebra.
2, 1pt) Deduce that $\operatorname{gr}_I A$ is Noetherian.

Recall the completion $\hat{A}$ of A constructed from the ideal I, Problem 3 of Homework 1. The goal of the next problem is to show $\hat{A}$ is Noetherian if A is. Note that this cannot be a consequence of the Hilbert basis theorem: $\hat{A}$ is not finitely generated as an A-algebra.

Problem 5, 5pts total. Assume A is Noetherian. Let J be an ideal of $\hat{A}$. For $j \geqslant 0$, let $\hat{I}_j$ denote the subset of $\hat{A}$ consisting of all sequences $(a_i)_{i=0}^{\infty}$ (where recall $a_i \in A/I^{i+1}$) such that $a_0 = \ldots = a_{j-1} = 0$.
1, 1pt) Prove that $\hat{I}_j$ is an ideal in $\hat{A}$ and identify $\hat{I}_j/\hat{I}_{j+1}$ with I^j/I^{j+1}.
Now consider $\operatorname{gr}_I J$ defined by

$$\operatorname{gr}_I J := \bigoplus_{i=0}^{\infty} (J \cap \hat{I}_i)/(J \cap \hat{I}_{i+1}).$$

2, 1pt) Produce an embedding $\operatorname{gr}_I J \hookrightarrow \operatorname{gr}_I A$ that realizes $\operatorname{gr}_I J$ is a homogeneous ideal in $\operatorname{gr}_I A$.
3, 2pts) Pick homogeneous generators $m_1, \ldots, m_k$ in $\operatorname{gr}_I J$. So each m_ℓ lies in a unique summand $(J \cap \hat{I}_{i_\ell})/(J \cap \hat{I}_{i_\ell+1})$. Let $\hat{m}_\ell$ denote some preimage of m_ℓ in $J \cap \hat{I}_{i_\ell}$. Prove that the elements $\hat{m}_1, \ldots, \hat{m}_\ell$ generate the ideal J in $\hat{A}$.
4, 1pt) Deduce that $\hat{A}$ is Noetherian.

The next problem gives an example of a $\mathbb{C}[x]$-module that is Artinian but not Noetherian.

Problem 6, 4pts total. Let $\mathbb{C}[x^{\pm 1}]$ denote the ring of Laurent polynomials, i.e. expressions $\sum_{i=-m}^{n} a_i x^i$ with natural addition and multiplication. Note that it contains $\mathbb{C}[x]$ as a subring, hence it is a $\mathbb{C}[x]$-module. Consider the quotient $\mathbb{C}[x]$-module $M := \mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$.

1, 2pts) Describe all possible submodules of M.

2, 1pt) Prove that M is an Artinian $\mathbb{C}[x]$-module.

3, 1pt) Prove that M is not Noetherian.

Problem 7, 4pts total. Let M be an A-module and $\psi : M \to M$ be an A-linear map.

1, 1pt) Suppose that M is Noetherian and ψ is surjective. Prove that ψ is an isomorphism. *Hint: consider a sequence of submodules* $\ker \psi^n \subset M$.

2, 1pt) Suppose that M is Artinian and ψ is injective. Prove that ψ is an isomorphism.

3, 2pt) Now suppose that M has finite length and $\psi : M \to M$ is an arbitrary A-linear map. Define the submodules $M_0 := \bigcup_n \ker \psi^n$ and $M_1 := \bigcap_n \operatorname{im} \psi^n$. Show that $M = M_0 \oplus M_1$. Furthermore, prove that M_0, M_1 are ψ-stable, the restriction $\psi|_{M_1}$ is an isomorphism $M_1 \to M_1$, while there is $d \in \mathbb{Z}_{>0}$ such that $(\psi|_{M_0})^d = 0$.

Problem 8, 2pts. Let $\mathbb{F}$ be a field. Classify the finitely generated modules over the formal power series ring $\mathbb{F}[[x]]$.

Problem 9, 2pts. Let A be a PID and M be a finitely generated A-module. So we have $M \cong A^{\oplus k} \oplus \bigoplus_{i=1}^{\ell} A/(p_i^{d_i})$. In terms of this decomposition, find a necessary and sufficient condition for M to be finite length.

Finally, we show that the condition of being finitely generated is essential for the main theorem on modules over PID's.

Problem 10, 3pts. Consider $\mathbb{Q}$ as an abelian group, i.e., a $\mathbb{Z}$-module. Prove that it is not isomorphic to any (incl. infinite) direct sum of copies of $\mathbb{Z}$ and cyclic groups.