MATH 380, HOMEWORK 3, DUE OCT 19

There are 10 problems worth 32 points total. Your score for this homework is the minimum of the sum of the points you've got and 24. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1,2. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative, unless stated otherwise, and contain 1.

Problem 1, 4pts. Consider the ring $\mathbb{C}[x,y]/(xy)$. Construct an isomorphism of its localization $(\mathbb{C}[x,y]/(xy))_{x+y}$ with the direct product $\mathbb{C}[x^{\pm 1}]\times\mathbb{C}[y^{\pm 1}]$.

Problem 2, 4pts total. *The ring $\mathbb{Z}[\sqrt{-5}]$ strikes back!* Consider the ring $A=\mathbb{Z}[\sqrt{-5}]$ and its ideal $I=(2,1+\sqrt{-5})$. Prove that the following localizations of I are rank one free modules over the corresponding localizations of A.

a, 2pts) I_2 over A_2.

b, 2pts) I_3 over A_3.

Problem 3, 3pts. *Prime ideals and localizations at complements of prime ideals.* Let $\mathfrak{p}$ be a prime ideal in A. Show that the map $\mathfrak{q}\mapsto\mathfrak{q}_{\mathfrak{p}}$ defines a bijection between the set of prime ideals of A contained in $\mathfrak{p}$ and the set of all prime ideals in $A_{\mathfrak{p}}$.

Problem 4, 2pts. *Direct sums vs localizations.* Let M,N be A-modules and S be a localizable subset of A. Produce a natural isomorphism $(M\oplus N)_S\xrightarrow{\sim}M_S\oplus N_S$.

Problem 5, 4pts total. *Hom modules and localizations.* Let M,N be A-modules and S be a localizable subset in A.

1, 1pt) Prove that the map $\psi\mapsto\psi_S:\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_{A_S}(M_S,N_S)$ is A-linear.

2, 2pts) Assume that A is Noetherian and M is finitely generated. Prove that the A-linear map from part 1 factors into the composition of the natural homomorphism $\iota:\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_A(M,N)_S$ and an isomorphism $\operatorname{Hom}_A(M,N)_S\xrightarrow{\sim}\operatorname{Hom}_{A_S}(M_S,N_S)$.

3, 1pt) Let N' be an A_S-module. Produce a natural isomorphism $\operatorname{Hom}_{A_S}(M_S,N')\xrightarrow{\sim}\operatorname{Hom}_A(M,N')$ of A-modules.

Problem 6, 4pts total. *Algebraic constructions as functors.* Show that the following constructions are naturally functors (*each part is 1pt, in parts a-c you are responsible for explaining what the functor does on morphisms*).

a) Sending a set I to the ring of polynomials $\mathbb{Z}[x_i]_{i\in I}$ *(in fact, we can replace $\mathbb{Z}$ with any ring)* is a functor $\mathsf{Sets} \to \mathsf{CommRings}$.

b) Sending a ring R to the group $R^\times$ of the invertible elements in R is a functor $\mathsf{Rings} \to \mathsf{Groups}$.

c) Sending a group G to its group ring $\mathbb{Z}G$ (a free modules with basis e_g labelled by elements of G and multiplication uniquely determined by $e_g e_h = e_{gh}$) is a functor from $\mathsf{Groups} \to \mathsf{Rings}$.

d) Sending a vector space V over $\mathbb{F}$ to its dual $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and sending a linear map $A : U \to V$ to the unique linear map $A^* : V^* \to U^*$ such that $[A^*\alpha](u) := \alpha(Au)$ ($\forall u \in U, \alpha \in V^*$) gives a functor $\mathbb{F}\text{-Vect} \to \mathbb{F}\text{-Vect}^{opp}$.

Problem 7, 2pts. (*Double dual*). Consider the full subcategory $\mathbb{F}\text{-Vect}_{fd}$ in $\mathbb{F}\text{-Vect}$ of all finite dimensional vector spaces. Prove that the endo-functor $\bullet^{**}$ of $\mathbb{F}\text{-Vect}_{fd}$ is isomorphic to the identity endo-functor.

Problem 8, 2pts total. *Functor morphisms for compositions of functors.* Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, $F, F' : \mathcal{D} \to \mathcal{C}$ and $G, G' : \mathcal{E} \to \mathcal{D}$ be functors, $\kappa : F \Rightarrow F'$ and $\eta : G \Rightarrow G'$ be functor morphisms.

1, 1pt) Explain how κ gives rise to functor morphisms $FG \Rightarrow F'G$ and $FG' \Rightarrow F'G'$ and how η gives rise to functor morphisms $FG \Rightarrow FG'$ and $F'G \Rightarrow F'G'$.

2, 1pt) Establish a commutative diagram involving the functors and functor morphisms from part 1.

Problem 9, 3pts total. *Naturality of the bijection in the proof of* $\mathrm{Hom}_{Fun}(F_X, F) \xrightarrow{\sim} F(X)$. Let X be an object in $\mathcal{C}$, F a functor $\mathcal{C} \to \mathsf{Sets}$ and F_X be the functor $\mathrm{Hom}_{\mathcal{C}}(X, \bullet) : \mathcal{C} \to \mathsf{Sets}$. Recall the bijection $\mathrm{Hom}_{Fun}(F_X, F) \xrightarrow{\sim} F(X)$, denote it by $\sigma_{X,F}$.

a, 1pt) Let G be another functor $\mathcal{C} \to \mathsf{Sets}$ and $\eta : F \Rightarrow G$ be a functor morphism. Prove that there is the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{Fun}(F_X, F) \arrow[r, "\sigma_{X,F}"] \arrow[d, "\eta\circ ?"] & F(X) \arrow[d, "\eta_X"] \\
\mathrm{Hom}_{Fun}(F_X, G) \arrow[r, "\sigma_{X,G}"] & G(X)
\end{tikzcd}$$

b, 2pt) Let Y be an object of $\mathcal{C}$ and $f : X \to Y$ be a morphism. Let f^* denote the corresponding element of $\mathrm{Hom}_{Fun}(F_Y, F_X)$. Prove that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}_{Fun}(F_X, F) \arrow[r, "\sigma_{X,F}"] \arrow[d, "? \circ f^*"] & F(X) \arrow[d, "F(f)"] \\
\mathrm{Hom}_{Fun}(F_Y, F) \arrow[r, "\sigma_{Y,F}"] & F(Y)
\end{tikzcd}$$

Problem 10, 4pts total. *Endomorphisms of functors.* Construct objects representing the forgetful functors F below and use this to determine the monoid $\mathrm{End}_{Fun}(F) = \mathrm{Hom}_{Fun}(F, F)$ (i.e. describe the set together with composition).

a, 2pts) $F : A\text{-Mod} \to \mathsf{Sets}$, where A is a commutative unital ring.

b, 2pts) $F : \mathsf{Rings} \to \mathsf{Sets}$.