MATH 380, HOMEWORK 4, DUE NOV 2

There are 8 problems worth 32 points total. Your score for this homework is the minimum of the sum of the points you've got and 24. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-3. The text in italic below is meant to to be comments to a problem but not a part of it.

Problem 1, 3pts total. *Product as a functor.* Let $\mathcal{C}$ be a category where every two objects have a product.

a, 2pts) Let $X, Y, X', Y' \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Hom}_{\mathcal{C}}(X, X'), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$. Construct a morphism $f \times g$ from $X \times Y$ to $X' \times Y'$.

b, 1pt) Explain how the product gives a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Problem 2, 4pts + bonus. *Adjoint functors.*

a, 2pts) Show that the polynomial ring functor $\mathsf{Sets} \to \mathsf{CommRings}$ from a) of Problem 6 in HW3 is left adjoint to the forgetful functor $\mathsf{CommRings} \to \mathsf{Sets}$.

b, 2pts) Show that the invertible elements functor from b) of Problem 6 in HW3 is right adjoint to the group ring functor from c) of that problem.

c, 0pts) Show that the inclusion functor $\mathbb{Z}\text{-Mod} \hookrightarrow \mathsf{Groups}$ *(that, as we have seen in Lecture 12 has a left adjoint functor, the abelinization)* has no right adjoint functor.

Problem 3, 3pts. *Compositions of adjoint functors.* Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}, F' : \mathcal{D} \to \mathcal{E}, G' : \mathcal{E} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that F is left adjoint to G and F' is left adjoint to G'. Prove that $F'F$ is left adjoint to GG'.

Problem 4, 4pts. *Endomorphisms of adjoint functors.* Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with bijection

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

1, 2pts) Let τ be a functor endomorphism of G. Let $\tau_{X,Y}$ denote the map

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y)), \psi \mapsto \tau_Y \circ \psi.$$

and let $\tau'_{X,Y} := \eta^{-1}_{X,Y} \circ \tau_{X,Y} \circ \eta_{X,Y}$ denote the corresponding map

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

Show that there is a unique functor endomorphism τ' of F such that $\tau'_{X,Y}$ sends $\varphi \in \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ to $\varphi \circ \tau'_X$.

2, 2pts) Consider the monoids $\mathrm{End}_{Fun}(F)$ and $\mathrm{End}_{Fun}(G)^{opp}$, where in the latter we reverse the order of multiplication. Establish a monoid isomorphism $\mathrm{End}_{Fun}(F) \xrightarrow{\sim} \mathrm{End}_{Fun}(G)^{opp}$.

Problem 5, 3pts. Let F be an additive functor $A\text{-Mod} \to B\text{-Mod}$. Show that, for any A-modules M_1, M_2, the B-modules $F(M_1 \oplus M_2)$ and $F(M_1) \oplus F(M_2)$ are naturally isomorphic. *The same conclusion holds for additive functors* $A\text{-Mod}^{opp} \to B\text{-Mod}$ *but you don't need to prove that.*

Problem 6, 4pts total. Let A be a (commutative unital) ring, $I \subset A$ be an ideal, M be an A-module.

1, 1pt) Identify $(A/I) \otimes_A M$ with M/IM.

2, 1pt) Construct a natural surjective A-linear map $I \otimes_A M \to IM$.

3, 1pt) Let $A = \mathbb{F}[x, y]$, where $\mathbb{F}$ is a field, $I = M = (x, y)$. Show that the A-linear map in 2) is not injective.

4, 1pt) *This is harder.* Moreover, show that the kernel of that map is finite dimensional over $\mathbb{F}$.

Problem 7, 5pts total. *Tensor algebra of a module. This problem discusses a left adjoint functor to the forgetful functor* A-Alg $\to A$-Mod. Let A be a commutative unital ring and M be an A-module. We write $M^{\otimes i}$ denote the i-fold tensor product $M \otimes_A M \otimes_A \ldots \otimes_A M$ (with $M^{\otimes 0} = A, M^{\otimes 1} = M$).

Consider the A-module $T_A(M) := \bigoplus_{i=0}^{\infty} M^{\otimes i}$. We define a graded algebra structure on $T_A(M)$ as follows: for $u \in M^{\otimes i}, v \in M^{\otimes j}$ their product is $u \otimes v$ in $M^{\otimes i} \otimes_A M^{\otimes j}$, which, as we know, is identified with $M^{\otimes(i+j)}$. This equips $T_A(M)$ with the structure of a graded associative A-algebra with unit ($1 \in A$). Note that it is not commutative.

1, 1pt) Let M be a free A-module with basis $x_i, i \in \mathcal{I}$. Identify $T_A(M)$ with the algebra $A\langle x_i \rangle_{i \in \mathcal{I}}$ of noncommutative polynomials in the variables x_i (recall that in that algebra we have a basis formed by words in the alphabet $x_i, i \in \mathcal{I}$, and the multiplication of basis elements is the concatenation of words).

2, 1pt) Let $\varphi : M \to N$ be an A-module homomorphism. Produce a graded algebra homomorphism $T_A(\varphi) : T_A(M) \to T_A(N)$.

3, 1pt) Show that T_A is a functor A-Mod $\to A$-Alg.

4, 2pts) Show that the functor T_A is left adjoint to the forgetful functor A-Alg $\to A$-Mod.

Problem 8, 6pts total. *Symmetric algebra of a module. This problem discusses a left adjoint functor to the forgetful functor* A-CommAlg $\to A$-Mod. Let A, M be as in the previous problem. Consider the two-sided ideal $I_M \subset T_A(M)$ generated by the elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1 \in T_A(M)$ for all $m_1, m_2 \in M$ (by definition, this means that I_M is the A-linear span of the elements of the form $\alpha(m_1 \otimes m_2 - m_2 \otimes m_1)\beta$ for $m_1, m_2 \in M, \alpha, \beta \in T_A(M)$). Set $S_A(M) := T_A(M)/I_M$.

1, 1pt) Show that $S_A(M)$ is a graded <u>commutative</u> A-algebra with the i-th graded component being the image of $M^{\otimes i} \subset S_A(M)$ in $S_A(M)$.

2, 1pt) Let $\varphi : M \to N$ be an A-module homomorphism. Produce a graded algebra homomorphism $S_A(\varphi) : S_A(M) \to S_A(N)$.

3, 1pt) Show that S_A is a functor A-Mod $\to A$-CommAlg.

4, 1pt) Show that the functor S_A is left adjoint to the forgetful functor A-CommAlg $\to$ A-Mod.

5, 1pt) Let M be a free module with basis $x_i, i \in \mathcal{I}$. Identify $S_A(M)$ with the algebra $A[x_i]_{i \in \mathcal{I}}$ of usual polynomials.

6, 1pt) Assume A is a Noetherian ring and M is a finitely generated A-module. Prove that $S_A(M)$ is a Noetherian ring.