MATH 380, HOMEWORK 5, DUE DEC 2

There are 11 problems worth 32 points total. Your score is the minimum of the number of points you get and 24. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-4. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are commutative and unital unless otherwise stated.

Problem 1, 3pts. *It's useful to look at objects representing functors...* Let A be a ring and M_1, M_2 be A-modules. Establish an A-algebra isomorphism $S_A(M_1 \oplus M_2) \cong S_A(M_1) \otimes_A S_A(M_2)$. *Symmetric algebras were introduced in Problem 8 of HW4.*

Problem 2, 3pts total. *Exactness and Homs.* Let M_1, M_2, M_3 be A-modules and let $\varphi_1 : M_1 \to M_2$ and $\varphi_2 : M_2 \to M_3$ be A-linear maps with $\varphi_2\varphi_1 = 0$.

1, 1pt) Prove that $0 \to M_1 \to M_2 \to M_3$ is exact if and only if the corresponding sequence

$$0 \to \mathrm{Hom}_A(N, M_1) \to \mathrm{Hom}_A(N, M_2) \to \mathrm{Hom}_A(N, M_3)$$

is exact for every A-module N.

2, 2pts) Prove that $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if the corresponding exact sequence

$$0 \to \mathrm{Hom}_A(M_3, N) \to \mathrm{Hom}_A(M_2, N) \to \mathrm{Hom}_A(M_1, N)$$

is exact for every A-module N.

Problem 3, 4pts total. *Adjointness implies opposite exactness.* Let A, B be rings. Let an additive functor $F : A\text{-Mod} \to B\text{-Mod}$ be left adjoint to an additive functor $G : B\text{-Mod} \to A\text{-Mod}$. Suppose that the bijections $\eta_{X,Y}$ in the definition of adjoint functors (Section 1.1 of Lecture 14) are abelian group homomorphisms.

1, 2pts) Show that F is right exact and G is left exact.

2, 2pts) Show that the following claims are equivalent:

(i) F sends projective modules to projective modules.
(ii) $F(A)$ is a projective B-module.
(iii) G is exact.

Problem 4, 3pts total. *This problem indicates why tensor product functors are ubiquitous.* Let A be a Noetherian ring. We write A-mod for the category of finitely generated A-modules. Let F be an additive functor $A\text{-mod} \to \mathbb{Z}\text{-Mod}$.

1, 1pt) Equip $F(A)$ with an A-module structure so that the underlying abelian group structure is the default structure on $F(A)$.

2, 2pts) Suppose that F is right exact. Establish an isomorphism of functors $F(A) \otimes_A \bullet$ (where the former is viewed as a functor to $\mathbb{Z}$-Mod by downgrading the A-module structure) and F.

There's a direct analog of 2) for left exact functors A-Mod$^{opp} \to \mathbb{Z}$-Mod. *And with some care you can replace* $\mathbb{Z}$-Mod *with categories of modules over more general rings.*

Problem 5, 3pts. *That ideal in* $\mathbb{Z}[\sqrt{-5}]$ *is here one more time*! Let $A = \mathbb{Z}[\sqrt{-5}]$ and I be the ideal $(2, 1+\sqrt{-5})$. Prove that $I \otimes_A I$ is isomorphic to A.

Problem 6, 2pts. Let K be a field and $A \subset K$ be a subring. Let B denote the integral closure of A in K. Prove that $B[x]$ is the integral closure of $A[x]$ in $K[x]$. *You can use the fact that the integral closure is a subring that is only proved under the Noetherian assumption in lectures.*

Problem 7, 3pts total. Let A be a domain and Γ be a subgroup in the group of ring automorphisms of A. Set $A^\Gamma := \{a \in A | \gamma a = a, \forall \gamma \in \Gamma\}$.
1, 1pt) Prove that A^Γ is a subring of A.
2, 1pt) Suppose A is normal. Prove that A^Γ is normal.
3, 1pt) Suppose Γ is finite. Prove that A is integral over A^Γ.

Problem 8, 4pts total + bonus. Prove that the normalizations of the following two $\mathbb{C}$-algebras are isomorphic to $\mathbb{C}[t]$.
a, 2pts) $\mathbb{C}[x, y]/(x^3 - y^2)$.
b, 2pts) $\mathbb{C}[x, y]/(x^2 + x^3 - y^2)$.
c – bonus – 0pts) Draw (the real loci of) the corresponding curves in $\mathbb{C}^2$. What do you see?

Problem 9, 2pts. Let A be a commutative ring and $S \subset A$ be a localizable subset. Prove that if A contains no nonzero nilpotent elements, then neither does A_S.

Problem 10, 3pts total. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ and I be an ideal in A. Set $J = \sqrt{I}$.
1, 2pts) Prove that there is $n > 0$ such that $J^n \subset I$.
2, 1pt) Use part (1) to conclude that the following two claims are equivalent:
(a) $V(I) \subset \mathbb{C}^n$ is finite.
(b) I has finite codimension in A.

Problem 11, 2pts. Let $\mathbb{F}$ be an infinite (for simplicity) field. Let M be an $\mathbb{F}[x_1, \ldots, x_n]$-module. Prove that the following two claims are equivalent:
(a) M is finite dimensional over $\mathbb{F}$.
(b) M has finite length as a module over $\mathbb{F}[x_1, \ldots, x_n]$.