MATH 380 FINAL, DUE DEC 16

There are four problems, worth 10 points each. For problems with two parts each part is worth 5 points. In your solution, you can refer to the lecture notes, but not to homework (with exception of solutions written by the instructor). You are not allowed to collaborate with other students in this class or seek external help. It is your responsibility to make sure that your solutions are detailed enough. Partial credit is given.

Below all rings are commutative and unital.

Problem 1. Let A be a Noetherian domain. Pick a non-invertible element $f \in A$ and consider the ideal (f) generated by f. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal. Suppose that $\mathfrak{p} \subset (f)$. Show that $\mathfrak{p} = (f)$.

Problem 2. Let A be a domain and I, J be nonzero ideals in A. Let S be the localizable subset of all nonzero elements in A. Recall that $A \subset A_S$ in this case.

1) For $\psi \in \mathrm{Hom}_A(I, J)$, explain how to interpret ψ_S as an element of A_S.
2) Identify $\mathrm{Hom}_A(I, J)$ with $\{\alpha \in A_S | \alpha I \subset J\}$ (as sets).

Problem 3. Let A be a ring, P be a finitely generated projective A-module and M be some A-module.

1) Show that $P^\vee := \mathrm{Hom}_A(P, A)$ is also a finitely generated projective A-module.
2) Prove that there is a well-defined A-linear map $P^\vee \otimes_A M \to \mathrm{Hom}_A(P, M)$ sending $\alpha \otimes m$ with $\alpha \in P^\vee, m \in M$, to $[p \mapsto \alpha(p)m]$ and that this map is an A-module isomorphism.

Problem 4. Let A be a ring and B be an A-algebra. Assume that B is a domain and is a finitely generated A-module.

1) Let A be a field. Prove that B is a field.
2) Let $A = \mathbb{F}[x]$, where $\mathbb{F}$ is a field. Prove that every nonzero prime ideal in B is maximal.