MATH 380, HOMEWORK 1, DUE SEPT 19

There are 8 problems worth 28 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, such as Problem 1, you can use previous parts to prove subsequent ones and get the corresponding credit. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 3pts total. *This problem defines the inverse limit of rings and studies its properties.*

Let $B_i, i \in \mathbb{Z}_{>0}$, be a collection of rings and, for $i < j$, let φ_{ij} be a homomorphism $B_j \to B_i$. Suppose that, for all $i < j < k$, we have $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$. Consider the infinite product $\prod_{i>0} B_i$ (the set of sequences $(b_i), i > 0$, where $b_i \in B_i$, with componentwise operations) and the subset in $\prod_{i>0} B_i$ consisting of all sequences (b_i) such that $b_i = \varphi_{ij}(b_j)$ for all $i < j$.

a, 1pt) Prove that this subset is a subring of $\prod_{i>0} B_i$. It is called the inverse limit of the sequence of the rings B_i and is denoted by $\varprojlim B_i$.

b, 1pt) Prove that, for each $j > 0$, the map $\varphi_j : \varprojlim B_i \to B_j$ defined by $(b_i) \mapsto b_j$ is a ring homomorphism satisfying $\varphi_j = \varphi_{jk} \circ \varphi_k$ for all $j < k$.

c, 1pt) *This is the universal property of the inverse limit $\varprojlim B_i$ and the homomorphisms φ_j.* Let A be a ring equipped with homomorphisms $\psi_i : A \to B_i$ satisfying $\psi_j = \varphi_{jk} \circ \psi_k$ for all $j < k$. Show that there exists a unique ring homomorphism $\psi : A \to \varprojlim B_i$ such that $\psi_j = \varphi_j \circ \psi$ for all j.

Problem 2, 4pts. *This problem describes a special case of the inverse limit of rings, the completion of a ring with respect to an ideal.*

Let A be a ring and I be its ideal. Set $B_j := A/I^j$ and let $\varphi_{jk} : A/I^k \to A/I^j$ for $j < k$ be the natural epimorphism. Persuade yourselves (not for credit) that this collection satisfies the assumptions of Problem 2. Set $\hat{A} := \varprojlim A/I^i$. This is the completion of interest.

a, 1pt) Prove that the homomorphisms $\varphi_j : \hat{A} \to A/I^j$ are surjective (for all $j > 0$).

An important example: when $A = \mathbb{Z}$ and $I = (p)$, where p is prime, the ring $\hat{A}$ is the ring of p-adic integers, it plays an important role in Algebraic Number theory.

b, 1pt) *We will concentrate on another important example: the ring of formal power series.* Let $A = B[x]$, where B is another ring, and $I = (x)$. Show that an element of $\hat{A}$ can be uniquely represented by a "formal power series", a sum $\sum_{i=0}^{\infty} b_i x^i$, where $b_i \in B$ (unlike with polynomials, we do not require that the sum is finite). Write formulas for the sum and product of two formal power series $\sum_{i=0}^{\infty} a_i x^i$ and $\sum_{i=0}^{\infty} b_i x^i$ in $\hat{A}$. *For $B = \mathbb{R}$ or $\mathbb{C}$, power series should be familiar from Calculus or Real/Complex Analysis. Unlike there, we do not require our power series to converge anywhere – which is why they are called formal.* The common notation for the ring of formal power series $\hat{A}$ is $B[[x]]$.

c, 1pt) *The ring of formal power series is closely related to the ring of polynomials. But it behaves differently, in fact, in many respects, it is simpler. The same applies to the p-adic* *integers vs the integers. In this part we discuss invertible elements in $B[[x]]$.* Prove that $\sum_{i=0}^{\infty} b_i x^i$ is invertible in $B[[x]]$ if and only if b_0 is invertible in B.

d, 1pt) Let B be a field. Describe all possible ideals in $B[[x]]$.

Why the name completion? This is a special case of the completion of a topological (abelian) group. A related procedure is used to get $\mathbb{R}$ from $\mathbb{Q}$. Namely, we have a topology on A, where the ideals I^j by definition form a base of neighborhoods of zero. For example, we can define the limit of a sequence (a_i) to be $a \in A$ if for all $j > 0$ there is $n > 0$ with $a_i - a \in I^j$ for all $i > n$. We can define the notion of a Cauchy sequence in A in a similar fashion. There is the usual equivalence relation on the set of Cauchy sequences. The ring $\hat{A}$ is identified with the set of equivalence classes.

Problem 3, 3pts. Let $\mathbb{F}$ be a field. Consider the ring A whose elements are formal sums of the form $\sum_{i=1}^{n} a_i x^{\lambda_i}$, where $a_1, \ldots, a_n \in \mathbb{F}$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ are nonnegative rational numbers, with natural addition and multiplication, i.e., $x^\lambda x^\mu := x^{\lambda+\mu}$ (informally, A is like the ring of polynomials but we allow fractional powers). Prove that:

a, 2pts) Every finitely generated ideal in A is principal. *Hint: use that every ideal in the ring of usual polynomials is principal.*

b, 1pt) The ideal of all elements, where the coefficient of x^0 is zero, is not finitely generated.

Problem 4, 5pts total. Let $\varphi : A \to B$ be a ring homomorphism and let J be an ideal in B. Set $I := \varphi^{-1}(J)$.

a, 1pt) Prove I is an ideal in A.

b, 1pt) Let J be prime. Is it always true that I is prime?

c, 1pt) Let J be maximal. Is it always true that I is maximal?

d, 1pt) Is it always true that $B\varphi(I) \subset J$?

e, 1pt) Is it always true that $J \subset B\varphi(I)$?

If you think a statement is true, provide a proof. If you think it is false, provide a counterexample.

Problem 5, 2pts total. Let A be a ring, M be an A-module, and $m \in M$.

1, 1pt) We define the subset $\operatorname{Ann}_A(m) := \{a \in A | am = 0\} \subset A$ ("Ann" stands for the "annihilator"). Prove that $\operatorname{Ann}_A(m)$ is an ideal in A.

2, 1pt) We define the subset $\operatorname{Ann}_A(M) := \{a \in A | am = 0, \forall m \in M\}$. Assume that M is generated by elements $m_1, \ldots, m_k \in M$. Prove that $\operatorname{Ann}_A(M) = \bigcap_{i=1}^{k} \operatorname{Ann}_A(m_i)$.

The next problem ultimately provides an important tool to compute Hom modules in some way.

Problem 6, 4pts total. Let L, M, N be A-modules. Then we have the composition map $\operatorname{Hom}_A(L, M) \times \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(L, N), (\varphi, \psi) \mapsto \psi \circ \varphi$.

a, 1pt) Prove that the composition map is A-bilinear, i.e., if we fix one of the arguments φ, ψ, then we get an A-linear map in the other argument.

Now consider four A-modules, M_1, M_2, M_3, N. Suppose we have A-linear maps $\varphi_1 : M_1 \to M_2, \varphi_2 : M_2 \to M_3$. Suppose that φ_2 is surjective, while $\operatorname{im} \varphi_1 = \ker \varphi_2$. Consider the maps, linear by part (a),

$$\tilde{\varphi}_1 : \mathrm{Hom}_A(M_2, N) \to \mathrm{Hom}_A(M_1, N), \psi_1 \mapsto \psi_1 \circ \varphi_1,$$
$$\tilde{\varphi}_2 : \mathrm{Hom}_A(M_3, N) \to \mathrm{Hom}_A(M_2, N), \psi_2 \mapsto \psi_2 \circ \varphi_2.$$

b, 1pt) Prove that $\tilde{\varphi}_2$ is injective.

c, 1pt) Prove that $\operatorname{im} \tilde{\varphi}_2 = \ker \tilde{\varphi}_1$.

d, 1pt) Suppose that, in the previous notation, $M_1 = A^{\oplus k}, M_2 = A^{\oplus \ell}$. So the map φ_1 is the multiplication by a matrix, denote it by $T = (t_{ji}), j = 1, \ldots, \ell, i = 1, \ldots, k$. Construct an isomorphism of A-modules between $\mathrm{Hom}_A(M_3, N)$ and the submodule of $N^{\oplus \ell}$ consisting of all ℓ-tuples $(n_1, \ldots, n_\ell)$ such that $\sum_{j=1}^{\ell} t_{ji} n_j = 0$ for all $i = 1, \ldots, k$.

Problem 7, 4pts total. *The goal of this problem is to produce an example of a projective module which is not free – and is even weirder than that!* Let $A = \mathbb{Z}[\sqrt{-5}]$. Consider the ideal $I := (2, 1 + \sqrt{-5})$.

a, 1pt) Show that I is not a free A-module.

b, 1pt) Consider the A-module epimorphism $\pi : A^{\oplus 2} \mapsto I$ given by $(\alpha, \beta) \mapsto 2\alpha + (1 + \sqrt{-5})\beta$. Identify $\ker \pi$ with I as A-modules.

c, 1pt) Prove that there is an A-module homomorphism $\iota : I \to A^{\oplus 2}$ such that $\pi \circ \iota = \mathrm{id}$. *Hint: realize ι in the form $z \mapsto (\gamma z, \delta z)$, where γ, δ are suitable elements of $\mathbb{Q}[\sqrt{-5}]$.*

d, 1pt) Prove that $A^{\oplus 2} = \ker \pi \oplus \operatorname{im} \iota$. Deduce an isomorphism $I^{\oplus 2} \cong A^{\oplus 2}$. *So $I^{\oplus 2} \cong A^{\oplus 2}$ but $I \ncong A$. This clearly cannot happen for vector spaces or abelian groups!*

In particular, I is projective but not free.

Problem 8, 3pts total. *If we change A and I "slightly", then the conclusions of the previous problem no longer hold.* Now let $A = \mathbb{Z}[x]$. Consider the ideal $I = (2, x)$. We still have the epimorphism $\pi : A^{\oplus 2} \twoheadrightarrow I$ given by $(f, g) \mapsto 2f + xg$.

a, 2pt) Show that the map $A \to \mathrm{Hom}_A(I, A)$ that sends $a \in A$ to m_a defined by $m_a(b) := ab$ is an A-linear isomorphism. *Hint: produce an isomorphism between the two modules using the conclusion of Problem 6 combined with the fact that $\mathbb{Z}[x]$ is a UFD, then prove that the isomorphism has the required form.*

b, 1pt) Prove that there is no A-module homomorphism $\iota : I \to A^{\oplus 2}$ such that $\pi \circ \iota = \mathrm{id}$. Deduce that I is not projective.

One could ask: what is the difference between the two rings in Problems 7 and 8. One difference is that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, while $\mathbb{Z}[x]$ is. But also these rings have different "dimensions": for $\mathbb{Z}[\sqrt{-5}]$ it is 1, and for $\mathbb{Z}[x]$ it is 2. The discussion of dimensions of Noetherian rings goes beyond the scope of this class.