MATH 380, HOMEWORK 3, DUE OCT 17

There are 6 problems worth 28 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if a problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to problems in Homeworks 1,2 (unless it is specified otherwise). The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 6pts total. *This problem is related to localization of quotient rings. The general setup is as follows.*

1, 2pts) Let A be a ring, $S \subset A$ be a multiplicative subset. Let $I \subset A$ be an ideal, and let π denote the projection $A \twoheadrightarrow A/I$. Note that $I[S^{-1}]$ is an ideal in $A[S^{-1}]$, while $\pi(S) \subset A/I$ is a multiplicative subset, *you don't need to prove this.* Establish a natural isomorphism between $A[S^{-1}]/I[S^{-1}]$ and $(A/I)[\pi(S)^{-1}]$.

The next two parts consider special cases of this situation.

2, 2pts) Let $A = \mathbb{C}[x, y]/(xy)$. We write $\bar{x}, \bar{y}$ for $x + (xy), y + (xy) \in A$. Set $S := \{\bar{x}^n | n \geqslant 0\}, I = (\bar{y})$. Establish an isomorphism between $A[S^{-1}]$ and the ring of Laurent polynomials $\mathbb{C}[x^{\pm 1}]$.

3, 2pts) Let still $A = \mathbb{C}[x, y]/(xy)$ but now $S = \{(\bar{x} + \bar{y})^n | n \geqslant 0\}$. Construct a ring isomorphism

$$A[S^{-1}] \xrightarrow{\sim} \mathbb{C}[x^{\pm 1}] \times \mathbb{C}[y^{\pm 1}].$$

A hint: Construct a homomorphism from the l.h.s. to the r.h.s. by first answering a question of how homomorphisms to direct products look like. Then show that this homomorphism is injective and surjective.

A geometric picture: the locus $\{(x, y)|xy = 0, x+y \neq 0\}$ is the disjoint union of $\{(x, 0)|x \neq 0\}$ and $\{(0, y)|y \neq 0\}$. We may revisit this in our discussion of connections to Algebraic geometry later in the class.

Problem 2, 7pts total. *Hom modules vs localizations, and how this helps to compute the modules of homomorphisms.* Let M, N be A-modules and S be a multiplicative subset in A.

1, 1pt) Prove that the map $\psi \mapsto \psi[S^{-1}] : \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{A[S^{-1}]}(M[S^{-1}], N[S^{-1}])$ is A-linear.

The following two parts are harder.

2, 2pts) Suppose that M is <u>finitely presented</u> meaning that there are k, ℓ and an A-linear map $\varphi : A^{\oplus k} \to A^{\oplus \ell}$ such that $M \cong A^{\oplus \ell}/\operatorname{im}\varphi$. Prove that the A-linear map from part 1) factors into the composition of the natural homomorphism $\iota_{\operatorname{Hom}_A(M,N)} : \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N)[S^{-1}]$ and an isomorphism $\operatorname{Hom}_A(M, N)[S^{-1}] \xrightarrow{\sim} \operatorname{Hom}_{A[S^{-1}]}(M[S^{-1}], N[S^{-1}])$.

Hint: something from HW1 should help...

3, 2pts) Now suppose that A is a Noetherian domain and $I \subset A$ is an ideal. Use the previous two parts to produce an A-module embedding of $\mathrm{Hom}_A(I, A)$ into $\mathrm{Frac}(A)$ whose image is $\{\alpha \in \mathrm{Frac}(A) | \alpha I \subset A\}$.

4, 1pt) Use part 3) to reprove a claim of Problem 8 in Homework 1: for the ideal $I = (2, x) \subset A = \mathbb{Z}[x]$, we have an isomorphism of A-modules $\mathrm{Hom}_A(I, A) \cong A$ (do not refer to your HW1 solution).

5, 1pt) Let $A = \mathbb{Z}[\sqrt{-5}], I = (2, 1 + \sqrt{-5})$. Use part 3) to re-establish an A-module isomorphism $\mathrm{Hom}_A(I, A) \cong I$ from Problem 7 in HW1 (do not refer to your HW1 solution).

Problem 3, 4pts. *Prime ideals and localizations at complements of prime ideals.* Let A be a commutative ring and $\mathfrak{p}$ a prime ideal in A. Show that the map $\mathfrak{q} \mapsto \mathfrak{q}_\mathfrak{p}$ defines a bijection between

- the set of prime ideals of A contained in $\mathfrak{p}$,
- and the set of all prime ideals in $A_\mathfrak{p}$.

The following problem introduces "locally free" modules over rings, the official definition will be given in the class later. Also later we will see that, in a reasonable class of modules, being locally free is equivalent to being projective. Suppose A is a ring, and let M be a finitely generated A-module. Consider the following conditions

(*) There are elements $f_1, \ldots, f_k \in A$ such that $(f_1, \ldots, f_k) = A$ and $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$-module for every $i = 1, \ldots, k$.

(**) For every maximal ideal $\mathfrak{m} \subset A$, the $A_\mathfrak{m}$-module $M_\mathfrak{m}$ is free.

It is the condition (*) that is used to relate projective modules to Algebraic geometry. This may be discussed as a bonus later in the class.

Problem 4, 7pts. 1, 1pt) *First, an example.* Consider the ring $A = \mathbb{Z}[\sqrt{-5}]$ and its ideal $I = (2, 1 + \sqrt{-5})$. Prove that the A-module $M := I$ satisfies (*) for $k = 2, f_1 = 2, f_2 = 3$.

We proceed by showing that (*) implies (**). The first two parts have to do with transitivity of localization, an important property on its own.

2,1pt) Now we get back to the general situation, where A is a commutative ring. Let $S \subset \tilde{S} \subset A$ be multiplicative subsets. Establish a natural isomorphism of rings $A[\tilde{S}^{-1}] \xrightarrow{\sim} A[S^{-1}][\iota(\tilde{S})^{-1}]$, where $\iota : A \to A[S^{-1}]$ is the natural homomorphism. *We use this isomorphism to identify these two rings.*

3, 1pt) Let M be an A-module. Establish an isomorphism $M[\tilde{S}^{-1}] \xrightarrow{\sim} M[S^{-1}][\iota(\tilde{S})^{-1}]$ of modules over the ring in part 1).

4, 1pt) Show that (*) implies (**).

Now we show that if A is Noetherian, then (**) implies (*).

5,1pt) Without the Noetherian assumption on A, prove that if M is a finitely generated module, and S is a multiplicative subset of A such that $M[S^{-1}] = \{0\}$, then there is $s \in S$ such that $M[s^{-1}] = \{0\}$.

6,1pt) Until the end of the problem suppose that A is Noetherian. Let M, N be finitely generated A-modules and $\psi : M \to N$ be an A-linear map. Suppose that S is a multiplicative subset such that $\psi[S^{-1}] : M[S^{-1}] \to N[S^{-1}]$ is an isomorphism. Show that there is $s \in S$ such that $\psi[s^{-1}] : M[s^{-1}] \to N[s^{-1}]$ is an isomorphism. *Hint: you should reduce this to the previous part.*

7, 1pt) *This is harder.* Prove that (**) implies (*).

The next two problems deal with functors from the category of sets (or its opposite) to the category of commutative algebras.

Problem 5, 2pts. Let $\mathbb{F}$ be a field. To a set X we assign the algebra $\mathbb{F}[X]$ of all functions $X \to \mathbb{F}$ with pointwise addition and multiplication. To a map of sets $\varphi : X \to Y$ we assign a map $\varphi^* : \mathbb{F}[Y] \to \mathbb{F}[X]$ given by $[\varphi^*(g)](x) = g(\varphi(x))$ for all $g \in \mathbb{F}[Y]$ and $x \in X$.

Check that $X \mapsto \mathbb{F}[X]$ and $\varphi \mapsto \varphi^*$ defines a functor $\mathsf{Sets}^{opp} \to \mathbb{F}\text{-}\mathsf{CommAlg}$ (the category of commutative $\mathbb{F}$-algebras). *You are responsible for checking all relevant conditions and axioms.*

We may see this construction again in our discussion of connections to Algebraic geometry, most likely as a bonus.

Problem 6, 2pts. Sending a finite set I to the ring of polynomials $\mathbb{Z}[x_i]_{i \in I}$ gives a functor $\mathsf{FinSets} \to \mathsf{CommRings}$, here the source category is that of finite sets. *You are responsible for explaining what this functor does on the level of morphisms, but not for checking axioms.*