MATH 380, HOMEWORK 4, DUE NOV 2

There are 8 problems worth 28 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-3. The text in italic below is meant to to be comments to a problem but not a part of it.

Problem 1, 4pts total. *Representing objects and endomorphisms of functors.*

1, 2pts) Prove that the forgetful functor $\mathsf{Rings} \to \mathsf{Sets}$ is represented by the ring $\mathbb{Z}[x]$. Use this to compute the monoid of endomorphisms of this forgetful functor. *Your answer should identify this monoid as a set and describe the multiplication of the monoid. Hint: this is related to the "composition" of polynomials.*

2, 2pts) Let A be a commutative ring. Determine the object representing the forgetful functor A-Mod $\to \mathsf{Sets}$ and use this to compute the monoid of endomorphisms of this functor.

Problem 2, 3pts. Let $\mathsf{FinGroups}$ denote the category of finite groups. Show that the coproduct $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ doesn't exist in $\mathsf{FinGroups}$. *Hint: the diherdral group D_n, i.e., the group of symmetries of the regular n-gon, has $2n$ elements and is generated by two elements of order 2 – you are welcome to use these facts in your solution without proof. And, in general, two non-trivial finite groups do not have the coproduct in* $\mathsf{FinGroups}$ *– but have it in* Groups.

The next two problems deal with formal properties of adjoint functors.

Problem 3, 3pts. *Compositions of adjoint functors.* Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}, F' : \mathcal{D} \to \mathcal{E}, G' : \mathcal{E} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that F is left adjoint to G and F' is left adjoint to G'. Prove that $F'F$ is left adjoint to GG'.

Problem 4, 4pts. *Endomorphisms of adjoint functors.* Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with bijection

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

1, 2pts) Let τ be a functor endomorphism of G. Let $\tau_{X,Y}$ denote the map

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y)), \psi \mapsto \tau_Y \circ \psi.$$

and let $\tau'_{X,Y} := \eta^{-1}_{X,Y} \circ \tau_{X,Y} \circ \eta_{X,Y}$ denote the corresponding map

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

Show that there is a unique functor endomorphism τ' of F such that $\tau'_{X,Y}$ sends $\varphi \in \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ to $\varphi \circ \tau'_X$. *Hint: Yoneda!*

2, 2pts) Consider the monoids $\mathrm{End}_{Fun}(F)$ and $\mathrm{End}_{Fun}(G)^{opp}$, where in the latter we reverse the order of multiplication. Establish a monoid isomorphism $\mathrm{End}_{Fun}(F) \xrightarrow{\sim} \mathrm{End}_{Fun}(G)^{opp}$.

The next three problems construct adjoint functors to some "boring" functors, inclusions and forgetful functors.

Problem 5, 3pts. Let $G : \mathsf{Rings} \to \mathsf{Monoids}$ be the forgetful functor.

1, 1pt) Show that the following constitutes a functor $F : \mathsf{Monoids} \to \mathsf{Rings}$. We consider a ring $\mathbb{Z}M$ that is a free abelian group with basis $e_m, m \in M$, and multiplication uniquely determined by $e_m e_{m'} = e_{mm'}$ (*known as the monoid ring*). We set $F(M) := \mathbb{Z}M$. Further, for a monoid homomorphism $\varphi : M \to N$, let $F(\varphi)$ be the unique ring homomorphism $\mathbb{Z}M \to \mathbb{Z}N$ with $[F(\varphi)](e_m) = e_{\varphi(m)}$. *You can assume that we indeed get associative rings and a ring homomorphism, but are responsible for checking functor axioms. Students who took 353 in the Spring, should find this construction familiar – this is a close relative of the group algebra that plays a crucial role in studying the representation theory of finite groups.*

2, 2pt) Show that F is left adjoint to G. *You are responsible for establishing the bijections η and checking the commutative diagrams.*

Problem 6, 3pts total. Let A be a commutative ring. Consider the category A-Alg of (associative and unital but not necessary commutative A-algebras) and its full subcategory A-CommAlg of commutative algebras. Let G be the inclusion functor A-CommAlg $\to$ A-Alg.

1, 1pt) Let B be an object of A-Alg. Consider the two-sided ideal $([B, B])$, the additive span of the elements of the form $b_1(bb' - b'b)b_2$ with $b_1, b_2, b, b' \in B$. Show that $\mathsf{Comm}(B) := B/([B, B])$ is a commutative A-algebra.

2, 1pt) Show that any A-algebra homomorphism $B_1 \to B_2$ descends to an algebra homomorphism $\mathsf{Comm}(B_1) \to \mathsf{Comm}(B_2)$. Use this to produce a *functor* $\mathsf{Comm} : A$-Alg $\to$ A-CommAlg. *You don't need to check the functor axioms but need to identify the data of a functor.*

3, 1pt) Show that $F := \mathsf{Comm}$ is left adjoint to G. *You are expected to produce the bijections η from the definition of an adjoint functor, but you don't need to verify the commutative diagrams.*

It turns out that G doesn't admit a right adjoint functor: the idea is that every algebra has a unique maximal commutative quotient, but does not have a unique maximal commutative subalgebra.

Problem 7, 4pts total. *Tensor and symmetric algebras of a module. This problem discusses left adjoint functors to the forgetful functor* A-Alg $\to$ A-Mod *and* A-CommAlg $\to$ A-Mod. Let A be a commutative ring and M be an A-module. Let $M^{\otimes i}$ denote the i-fold tensor product $M \otimes_A M \otimes_A \ldots \otimes_A M$ (with $M^{\otimes 0} = A, M^{\otimes 1} = M$).

Consider the A-module $T_A(M) := \bigoplus_{i=0}^{\infty} M^{\otimes i}$. We define a graded algebra structure on $T_A(M)$ as follows: for $u \in M^{\otimes i}, v \in M^{\otimes j}$ their product is $u \otimes v$ in $M^{\otimes i} \otimes_A M^{\otimes j}$, which, as we know, is identified with $M^{\otimes (i+j)}$. This equips $T_A(M)$ with the structure of a graded associative A-algebra with unit $1 \in A$. *Check this, not for credit. Recall that graded algebras were discussed in HW2. Note that $T_A(M)$ is not commutative.*

1, 1pt) Let $\varphi : M \to N$ be an A-module homomorphism. Produce a graded algebra homomorphism $T_A(\varphi) : T_A(M) \to T_A(N)$. Show that T_A is a functor A-Mod $\to$ A-Alg.

2, 1pt) Show that the functor T_A is left adjoint to the forgetful functor A-Alg $\to$ A-Mod. *You are expected to produce the bijections η from the definition of an adjoint functor, but you don't need to verify the commutative diagrams.*

3, 1pt) Define the *symmetric algebra* functor, S_A as the composition $\mathsf{Comm} \circ T_A$. Use approprite previous problems in this homework (or parts of them) to show that S_A is the left adjoint functor to the forgetful functor A-CommAlg $\to$ A-Mod.

4, 1pt) Let M be a free A-module with basis $x_1, \ldots, x_n$. We write $\mathcal{C}$ for the category A-CommAlg. Establish an isomorphism of functors $\mathcal{C} \to \mathsf{Sets}$

$$\mathrm{Hom}_{\mathcal{C}}(S_A(M), \bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A[x_1, \ldots, x_n], \bullet)$$

(*including checking the appropriate commutative diagram*). Deduce that there is an algebra isomorphism $S_A(M) \cong A[x_1, \ldots, x_n]$.

Problem 8, 4pts total. *And, finally, a problem on computing tensor products of modules.* Let A be a (commutative) ring, $I \subset A$ be an ideal, M be an A-module.

1, 1pt) Identify $(A/I) \otimes_A M$ with M/IM.

2, 1pt) Construct a natural surjective A-linear map $I \otimes_A M \to IM$.

3, 2pt) Let $A = \mathbb{C}[x, y]$, $I = M = (x, y)$. Show that the A-linear map in 2) is not injective.