MATH 380, HOMEWORK 5, DUE NOV 16

There are 7 problems worth 26 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if the problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to Homeworks 1-4. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are commutative.

Problem 1, 4pts. *Tensor products of finite fields.* Let p be a prime integer. For its power q we write $\mathbb{F}_q$ for the field with q elements, this is an $\mathbb{F}_p$-algebra. Pick positive integers k, ℓ and set $q = p^k, q' = p^\ell$.

1, 2pts) Assume k, ℓ are coprime. Prove that there is a ring isomorphism $\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_{q'} \cong \mathbb{F}_{p^{k\ell}}$ (*Hint: a certain relevant subset is simultaneously a vector space over* $\mathbb{F}_q$ *and a vector space over* $\mathbb{F}_{q'}$).

2, 2pts) Prove that there is a ring isomorphism $\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_q \cong \mathbb{F}_q^k$ (the direct product of k copies of $\mathbb{F}_q$, *hint: use an exercise from Lecture 17 notes and some facts from the Fields and Galois theory course*).

Problem 2, 4pts. *Additive functors and direct sums.* Let F be an additive functor A-Mod $\to$ B-Mod for commutative rings A, B. Show that for all A-modules M_1 and M_2, we have $F(M_1 \oplus M_2) \cong F(M_1) \oplus F(M_2)$, an isomorphism of B-modules. *Hint: look at the proof of the distributivity of the tensor product of modules in Lecture 16.*

Problem 3, 3pts total. *Exactness and Homs.* Let M_1, M_2, M_3 be A-modules and let $\varphi_1 : M_1 \to M_2$ and $\varphi_2 : M_2 \to M_3$ be A-linear maps with $\varphi_2\varphi_1 = 0$.

1, 1pt) Prove that the sequence $0 \to M_1 \to M_2 \to M_3$ is exact if and only if the corresponding sequence

$$0 \to \mathrm{Hom}_A(N, M_1) \to \mathrm{Hom}_A(N, M_2) \to \mathrm{Hom}_A(N, M_3)$$

is exact for every A-module N.

2, 2pts) Prove that the sequence $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if the corresponding sequence

$$0 \to \mathrm{Hom}_A(M_3, N) \to \mathrm{Hom}_A(M_2, N) \to \mathrm{Hom}_A(M_1, N)$$

is exact for every A-module N.

Problem 4, 4pts total. *This problem indicates why tensor product functors are ubiquitous.* Let A be a Noetherian ring. We write A-mod for the category of finitely generated A-modules. Let F be an additive functor $A\text{-mod} \to \mathbb{Z}\text{-Mod}$. Further, let For be the forgetful functor

$$A\text{-mod} \to \mathbb{Z}\text{-Mod}$$

1, 2pts) Equip $F(A)$ with an A-module structure so that the underlying abelian group structure is the default structure on $F(A)$, i.e., the structure coming from the fact that the target of F is $\mathbb{Z}$-Mod. *Hint: use that F is a functor.*

2, 2pts) *This is harder.* Suppose that F is right exact. Establish an isomorphism of functors $\mathsf{For}(F(A) \otimes_A \bullet)$ and F. *Hint: you may want to produce a functor isomorphism η first on the A-module A, then on the A-module $A^{\oplus n}$, then in general. The construction won't make it clear that what you get is a functor morphism, so a check is needed.*

There's a direct analog of 2) for left exact functors $A\text{-mod}^{opp} \to \mathbb{Z}\text{-Mod}$, these are Homs to some A-module. And with some care you can replace $\mathbb{Z}$-Mod with categories of modules over more general rings.

Problem 5, 4pts total. *Adjointness implies opposite exactness.* Let A, B be rings. Let an additive functor $F : A\text{-Mod} \to B\text{-Mod}$ be left adjoint to an additive functor $G : B\text{-Mod} \to A\text{-Mod}$.

1, 2pts) Show that F is right exact and G is left exact.

2, 2pts) Show that the following claims are equivalent:

(i) F sends projective modules to projective modules.
(ii) $F(A)$ is a projective B-module.
(iii) G is exact.

Problem 6, 4pts. *That ideal in $\mathbb{Z}[\sqrt{-5}]$ is here once more*! Let $A = \mathbb{Z}[\sqrt{-5}]$ and I be the ideal $(2, 1+\sqrt{-5})$. Prove that $I \otimes_A I \cong A$, an A-module isomorphism (*hint: something from the previous homeworks should help*).

Problem 7, 3pts. Let A be a local domain with maximal ideal $\mathfrak{m}$ and $\mathbf{k} := A/\mathfrak{m}$, the so called residue field. Show that a nonzero finitely generated ideal $I \subset A$ is principal if and only if $\dim_{\mathbf{k}} I/\mathfrak{m}I = 1$.