MATH 380, HOMEWORK 6, DUE DECEMBER 7

There are 6 problems worth 25 points total. Your score for this homework is the minimum of the sum of the points you've got and 20. Note that if a problem has several related parts, you can use previous parts to prove subsequent ones and get the corresponding credit. You can also use previous problems to solve subsequent ones and refer to your solutions to Homeworks 1-5 – or to the posted solutions. The text in italic below is meant to to be comments to a problem but not a part of it.

All rings are assumed to be commutative.

Problem 1, 4pts total. *The problem concerns rings of invariants and their properties.*

Let A be a domain and Γ be a subgroup in the group of ring automorphisms of A *(recall that an automorphism is an isomorphism $A \to A$; they do form a group).* Set $A^\Gamma := \{a \in A | \gamma a = a, \forall \gamma \in \Gamma\}$.

1, 1pt) Prove that A^Γ is a subring of A.

2, 1pt) Suppose A is normal. Prove that A^Γ is normal.

3, 1pt) Suppose Γ is finite. Prove that A is integral over A^Γ.

4, 1pt) Suppose B is a normal domain, $K := \mathrm{Frac}(B)$, L is a finite Galois extension of K with Galois group Γ, and A is the integral closure of B in L. Prove that $B = A^\Gamma$, the equality of subrings in L.

This provides an important construction of rings with nice properties: we start with $A = \mathbb{F}[x_1, \ldots, x_n]$ and, say, a finite group Γ acting by linear change of variables $x_1, \ldots, x_n$.

Problem 2, 4pts. *This problem concerns some formal properties of normalization.*

1, 2pts) Let A be a normal domain and $S \subset A$ be a multiplicative subset. Show that $A[S^{-1}]$ is a normal domain.

2, 2pts) Let L be a field, I be an index set, and $A_i \subset L, i \in I$, be subrings. Show that if all A_i are normal, then $\bigcap_{i \in I} A_i$ is normal.

Problem 3, 4pts. *This problem computes a normalization in an algebro-geometric setup.* Let $\mathbb{F}$ be a field and x, y be indeterminates.

1, 2pts) Show that $A := \mathbb{F}[x, y]/(x^2 - y^3)$ is a domain.

2, 2pts) Identify the normalization of A with $\mathbb{F}[t]$, where t is another indeterminate. *Hint: observe that $x/y \in \mathrm{Frac}(A)$ is integral over A.*

3, 0pts) Draw the graph of $x^2 = y^3$. What do you see?

Problem 4, 4pts. *This problem introduces a useful property and uses it to establish one of conditions of the Dedekind domains for the algebras of functions on planar curves. You don't need to know the definition of a Dedekind domain for this problem..*

1, 2pts) Let $A \subset B$ be domains such that B is integral over A. Let $I \subset B$ be a nonzero ideal. Show that $I \cap A \neq \{0\}$.

2, 2pts) Let $\mathbb{F}$ be a field, and $f \in \mathbb{F}[x, y]$ be an irreducible polynomial. Let $A := \mathbb{F}[x, y]/(f)$. Prove that every nonzero prime ideal in A is maximal. *Hint: use the Noether normalization lemma.*

Problem 5, 4pts total. Let $\mathbb{F}$ be an algebraically closed field. Let $A = \mathbb{C}[x_1, \ldots, x_n]$ and I be an ideal in A. Set $J = \sqrt{I}$.

1, 2pts) Prove that there is $n > 0$ such that $J^n \subset I$.

2, 2pts) Use part (1) to conclude that the following two claims are equivalent:

(a) $V(I) \subset \mathbb{F}^n$ is finite.

(b) I has finite codimension in A, i.e., $\dim_{\mathbb{F}} A/I < \infty$.

Problem 6, 5pts total. *Products of algebraic subsets.* Let $\mathbb{F}$ be an algebraically closed field, and $X \subset \mathbb{F}^n, Y \subset \mathbb{F}^m$ be algebraic subsets.

1, 1pt) Show that $X \times Y \subset \mathbb{F}^{n+m}$ is an algebraic subset.

2, 2pts) Show that $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y]$ has no nonzero nilpotent elements. *Hint: apply homomorphisms $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] \to \mathbb{F}[Y]$ evaluating at points of X.*

3, 2pts) Establish a natural isomorphism $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] \xrightarrow{\sim} \mathbb{F}[X \times Y]$.

The conclusion is that the tensor product of algebras of functions corresponds to the product of algebraic subsets. This may serve as a motivation to consider tensor products of algebras.