MATH 380 FINAL: DUE DECEMBER 17

There are seven problems worth 15 points each. Your official score for this final is the minimum of 80 and your total score. The penalty for late submissions is 10% of the official score per day (e.g. two day delay results in the penalty of 20%). The penalty can only be waived with a Dean excuse.

Collaboration is NOT allowed. You can only use the following:

- **the course notes (use the posted notes for references). This includes all exercises but excludes all Bonus parts (side remarks, bonus sections of the lectures, as well as the three bonus lectures B1-B3).**
- **statements of the homework problems: if the problem asks you to prove X, you can assume X holds in your solutions.**
- **The posted (in Announcements on Canvas) homework solutions.**

As usual, you can use the statements of previous problems in your solutions of the later ones (even if you haven't solved the previous problem), and also use the previous parts of a given problem to solve the later parts. If you wish to refer to your HW solutions, please include them into your submission of the final.

As usual, partial credit is given. Try to solve as much in a problem as you can.

Convention: in all problems below A denotes a commutative domain, which is not a field. M denotes a finitely generated A-module.

Problem 1. *Recall that any localization of A can be viewed as a subring in the fraction field* $\mathrm{Frac}(A)$*. This allows us to take intersections of localizations.*

1, 8pts) Let $f_1, \ldots, f_k \in A$ be such that $(f_1, \ldots, f_k) = A$. Show that for all $n > 0$ there are elements $a_1, \ldots, a_k \in A$ such that $\sum_{i=1}^k a_i f_i^n = 1$ and use this to deduce that $\bigcap_{i=1}^k A[f_i^{-1}] = A$.

2, 7pts) Assume that A is Noetherian. Show that $\bigcap_{\mathfrak{m}} A_{\mathfrak{m}} = A$, where the intersection is taken over all maximal ideals $\mathfrak{m} \subset A$.

Problem 2. Let A be Noetherian. Show that the following conditions are equivalent:

a) A is Dedekind,

b) The localization $A_{\mathfrak{m}}$ is Dedekind for all maximal ideals $\mathfrak{m} \subset A$.

Problem 3. Let A be Noetherian and local with maximal ideal $\mathfrak{m}$.

1, 2pts) Consider the A-subalgebra $\tilde{A}$ in $A[x]$ consisting of all polynomials $f(x) = \sum_{i=0}^n a_i x^i$ with $a_i \in \mathfrak{m}^i$ (where $\mathfrak{m}^0 = A$; *you don't need to check it is a subalgebra*). Show that $\tilde{A}$ is finitely generated (as an algebra) over A.

2, 2pts) Deduce that $\tilde{A}$ is Noetherian.

3, 2pts) Equip the abelian group $\tilde{M} := \bigoplus_{i=0}^{\infty} \mathfrak{m}^i M$ with a natural $\tilde{A}$-module structure (*hint: we can form the "polynomials" $M[x]$, then $\tilde{M} = \{\sum_{i=0}^{n} m_i x^i\}$, where $m_i \in \mathfrak{m}^i M$*). Show that $\tilde{M}$ is finitely generated over $\tilde{A}$ (as a module).

4, 3pts) Let $N \subset M$ be an A-submodule. Equip $\bigoplus_{i=0}^{\infty}(N \cap \mathfrak{m}^i M)$ with the structure an $\tilde{A}$-submodule of $\tilde{M}$, and show that $\bigoplus_{i=0}^{\infty}(N \cap \mathfrak{m}^i M)$ is finitely generated as an $\tilde{A}$-module.

5, 3pts) Show that for all $n > 0$, there is $k > n$ such that $\mathfrak{m}^n N \supset \mathfrak{m}^k M \cap N$.

6, 3pts) Now let N denote the ideal $\bigcap_{i=0}^{\infty} \mathfrak{m}^i$. Show that it is zero.

Problem 4. Let A be local and Noetherian. Let $\mathfrak{m}$ denote the maximal ideal of A. Prove that the following conditions are equivalent.

a) A is a PID.

b) A is Dedekind.

c) $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.

d) $\dim_{A/\mathfrak{m}} \mathfrak{m}^k/\mathfrak{m}^{k+1} = 1$ for all $k > 0$.

Problem 5. Let $\mathbb{F}$ be an algebraically closed field, x, y be indeterminates, and $f \in \mathbb{F}[x, y]$ be an irreducible polynomial. Set $A := \mathbb{F}[x, y]/(f)$. Show that the following conditions are equivalent:

a) A is Dedekind,

b) For any point $(a, b) \in \mathbb{F}^2$ with $f(a, b) = 0$, the gradient $(\partial f/\partial x, \partial f/\partial y)$ is nonzero at (a, b).

For the next two problems we need two definitions. The module M is called:

- *invertible if there is an A-module M' with $M \otimes_A M' \cong A$, an isomorphism of A-modules.*
- *torsion free if the equality $am = 0$ with $a \in A, m \in M$ implies $a = 0$ or $m = 0$.*

Problem 6. Assume A is Noetherian. Show that, for the statements below, a)⇒b)⇒c). *You may find the first implication harder to prove. Also one can remove the Noetherian assumption and also prove that M is flat, but you are not asked to do this.*

a) M is invertible (see the definition before the problem statement).

b) The unique A-linear map $I \otimes_A M \to M$ with $b \otimes m \mapsto bm$ is injective for every ideal $I \subset A$. *You don't need to justify its existence.*

c) M is torsion free.

Problem 7.

1, 5pts) For a commutative ring B, its multiplicative subset S, and B-modules L, N construct an isomorphism $(L \otimes_B N)[S^{-1}] \xrightarrow{\sim} L[S^{-1}] \otimes_{B[S^{-1}]} N[S^{-1}]$.

In the next two parts we assume A is a Dedekind domain.

2, 5pts) Prove that for fractional ideals $I, J \subset \mathrm{Frac}(A)$, we have $I \otimes_A J \cong IJ$, an isomorphism of A-modules.

3, 5pts) Prove that M is invertible if and only if M is isomorphic to a fractional ideal for A.